22nd Bay Area Mathematical Olympiad

BAMO-12 Exam

February 26, 2020

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

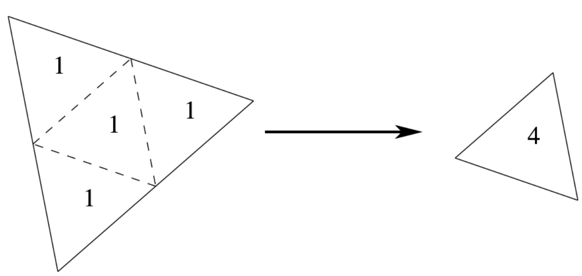
Problems 1 and 2 (in part) are on this side; problems 2 (in part), 3, 4, and 5 are on the other side.

1 Find all real numbers x that satisfy the equation

$$\frac{x-2020}{1}+\frac{x-2019}{2}+\cdots+\frac{x-2000}{21}=\frac{x-1}{2020}+\frac{x-2}{2019}+\cdots+\frac{x-21}{2000},$$

and simplify your answer(s) as much as possible. Justify your solution.

2 Consider a sheet of paper in the shape of an equilateral triangle creased along the dashed lines as in the figure below on the left. Folding over each of the three corners along the dashed lines creates a new object which is uniformly four layers thick, as in the figure below on the right. The number in each region indicates that region's thickness (in layers of paper).



We have just seen one example of how a plane figure can be folded into an object with a uniform thickness. This problem asks you to produce several other examples. In each case, you may fold along any lines. The different parts that are folded may or may not be congruent. Assume that paper may be folded any number of times without tearing or becoming too thick to fold. If needed, you can use any of the following tools:

- a magic ruler with which you can draw a line through any two given points and you can split any segment into as many equal parts as you wish; and
- a right triangle tool with which you can drop perpendiculars from points to lines and erect perpendiculars to lines from points on them.

(Problem 2 continues on reverse)

(Problem 2, continued from front)

Given these rules:

(a) Show how to fold an equilateral triangle into an object with a uniform thickness of 3 layers.

(b) Show how to fold a 30°-60°-90° triangle into an object with a uniform thickness of 3 layers.

(c) Show that every triangle can be folded into an object with a uniform thickness of 2020 layers.

3 The integer 202020 is a multiple of 91. For every positive integer n, show how n additional 2's may be inserted into the digits of 202020 so that the resulting $(n+6)$-digit integer is also a multiple of 91.

For example, a possible way to do this when $n = 5$ is $\underline{2}2020\underline{2}20\underline{222}$ (the inserted 2's are underlined).

4 Consider $\triangle ABC$. Choose a point M on its side BC and let O be the center of the circle passing through the vertices of $\triangle ABM$. Let k be the circle that passes through A and M and whose center lies on line BC. Let line MO intersect k again in point K. Prove that the line BK is the same for any choice of point M on segment BC, so long as all of these constructions are well-defined.

5 Let S be a set of $a+b+3$ points on a sphere, where a, b are nonnegative integers and no four points of S are coplanar (that is, no four points lie on a plane). Determine how many planes pass through three points of S and separate the remaining points into a points on one side of the plane and b points on the other side.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2020 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 2–4PM on Saturday, March 14. This event will include a mathematical talk by **Alon Amit (Intuit)**, refreshments, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor and/or `bamo.org` for a more detailed schedule, plus directions.