1st Bay Area Mathematical Olympiad

February 23, 1999

Problems and Official Solutions

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive much credit.

The five problems below are arranged in roughly increasing order of difficulty. In particular, problems 4 and 5 are quite difficult. We don't expect many students to solve all the problems; indeed, solving just one problem completely is a fine achievement. We do hope, however, that you find the experience of thinking deeply about mathematics for 4 hours to be a fun and rewarding challenge. We hope that you find BAMO interesting, and that you continue to think about the problems after the exam is over.

1. Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7.)[1]

Solution. One of the twelve numbers must be a multiple of twelve; call it $a = 12n$. Among the proper divisors of a are the integers $2n, 3n, 4n, 6n$. These sum to $15n > a$. □

2. Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0, a)$ and $B = (0, b)$ with $0 < a < b$. Let P be any other point on the circle, let Q be the intersection of the line through P and A with the x-axis, and let $O = (0, 0)$. Prove that $\angle BQP = \angle BOP$.

Solution. We make use of the fact that an angle inscribed in a circle has measure equal to one-half of the arc subtended. Since the x- and y-axes meet in a right angle, the circle C_1 through B, O, and Q has QB as a diameter. Also, $\angle APB$ is a right angle, since AB is the diameter of C. But this means that $\angle QPB$ and $\angle QOB$ are both right angles, so that P, O, Q, B all lie on circle C_1. Thus the two angles in question, $\angle BQP$ and $\angle BOP$, are inscribed in C_1, subtend the same arc, and are therefore equal. □

[1] Adapted from Moscow Math Olympiad, 1972

3. A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too (see diagram on p.1). Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)

Solution. The problem is solved if there is a way to change the orientation of any specified single key, without changing any of the others. This is equivalent to finding a way to switch the chosen key an *odd* number of times, while switching all other keys a *even* number of times.

This can be done by switching all keys on the same row and column of the chosen key (including the chosen key). To see why, choose a key K. If we switch it and all of its "sisters" that share the same row and column, K will be switched 7 times. Now, examine the other 15 keys in the lock. There are two cases: either the key is a sister of K, or not. Suppose that L is a sister of K, say, sharing a row with K. Then L will be switched 4 times. For the other case, suppose M is not a sister of K. Then M will be switched twice, because among the 6 sisters of K which are turned, exactly two of them share a row or column with M. Consequently, of all the keys in the lock, only K is switched an odd number of times. All other keys are switched either 2 or 4 times, leaving their orientation unchanged. Thus we will be able to open the lock by selecting each horizontal key one-by-one, and turning it and all of its sister keys. □

4. Finitely many cards are placed in two stacks, with more cards in the left stack than the right. Each card has one or more distinct names written on it, although different cards may share some names. For each name, we define a "shuffle" by moving every card that has this name written on it to the opposite stack. Prove that it is always possible to end up with more cards in the right stack by picking several distinct names, and doing in turn the shuffle corresponding to each name.[2]

Solution. Let the number of cards be c and let the number of distinct names be n. Each card contains a set of names; denote these sets by $S_1, S_2, \ldots, S_c$ (some of these sets may share elements). Now let E be a subset of the set of n names, and denote by $D(E)$ the difference of the number of cards in the left stack and the number of cards in the right stack, after the cards are shuffled by the names in E. We are given that $D(\emptyset) > 0$; we must show that there exists a subset E such that $D(E) < 0$.

Consider the sum of all $D(E)$ as E ranges through all 2^n possible subsets of names. If we can show that this sum is equal to zero, we will be done, since $D(\emptyset) > 0$ is just one term in this sum, forcing at least one other term to be negative. For $i = 1, 2, \ldots, c$, let $v_i = +1$ or -1 if the ith card is initially in the left or right stack, respectively. Then

$$D(\emptyset) = v_1 + v_2 + \cdots + v_c = \sum_{i=1}^{c} v_i.$$

[2]Adapted from USSR Math Olympiad, 1968.

Now consider what happens when we shuffle the cards corresponding to a subset E of names. The ith card will move back and forth from one stack to the other a total of $|E \cap S_i|$ times ($|A|$ means the number of elements in the set A). The only thing that matters is whether this value is even or odd. Thus we have

$$D(E) = \sum_{i=1}^{c} (-1)^{|E \cap S_i|} v_i.$$

It remains to sum this expression over all subsets E. Let us examine what happens just for one card; i.e., let us compute

$$\sum_E (-1)^{|E \cap S_i|} v_i$$

for a *fixed* i as E ranges over all 2^n subsets of names. If we can show that this equals zero, then the entire sum will equal zero and we are done. Let $|S_i| = k$. Then $|E \cap S_i|$ will range from 0 to k inclusive. For $0 < r < k$, how many subsets E are there such that $|E \cap S_i| = r$? There are $\binom{k}{r}$ subsets of S_i with r elements. Fix one of them, call it T. Then E must contain all the elements of T, plus any subset of the names that are not contained in S_i. In other words, there are 2^{n-k} subsets containing T, and thus there are $\binom{k}{r} 2^{n-k}$ subsets E altogether satisfying $|E \cap S_i| = r$. Therefore

$$\sum_E (-1)^{|E \cap S_i|} v_i = v_i \sum_{r=0}^{k} (-1)^r \binom{k}{r} 2^{n-k} = 2^{n-k} v_i \sum_{r=0}^{k} (-1)^r \binom{k}{r}.$$

By the binomial theorem, we have

$$2^{n-k} v_i \sum_{r=0}^{k} (-1)^r \binom{k}{r} = 2^{n-k} v_i (1-1)^k = 0. \quad \square$$

5. Let $ABCD$ be a cyclic quadrilateral (a quadrilateral which can be inscribed in a circle). Let E and F be variable points on the sides AB and CD, respectively, such that $AE/EB = CF/FD$. Let P be the point on the segment EF such that $PE/PF = AB/CD$. Prove that the ratio between the areas of triangle APD and BPC does not depend on the choice of E and F.[3]

Solution. There are two cases to consider. First, assume that the lines AD and BC are not parallel and meet at S. Since $ABCD$ is cyclic, $\triangle ASB$ and $\triangle CSD$ are similar. Since $AE/AB = CF/CD$, then $AE/CF = AB/CD = AS/CS$, and $\triangle ASE$ and $\triangle CSF$ are also similar ($\angle SAE = \angle SCD$). Therefore, $\angle DSE = \angle CSF$.

[3]Shortlisted for the International Math Olympiad, 1998.

By similarity, we have

$$\frac{SE}{SF} = \frac{SA}{SC} = \frac{AB}{CD} = \frac{PE}{PF},$$

which means that SP is the bisector of angle S in $\triangle FSE$. This implies that $\angle ESP = \angle FSP$ and hence $\angle ASP = \angle BSP$, so SP is also the bisector of angle S in $\triangle ASB$. This means that P is equidistant from the lines AD and BC. Thus

$$[APD]/[BPC] = AD/BC,$$

which is a constant (we use the notation $[ABC]$ for the area of $\triangle ABC$).

For the second case, assume that AD and BC are parallel. Then $ABCD$ is an isosceles trapezoid with $AB = CD$, and we have $BE = DF$. Let M and N be the midpoints of AB and CD, respectively. Then $ME = NF$ and E and F are equidistant from the line MN. Thus P, the midpoint of EF, lies on MN. This implies that P is equidistant from AD and BC, and hence

$$[APD]/[BPC] = AD/BC. \quad \square$$

You are cordially invited to attend the **BAMO 1999 Awards Ceremony**, which will be held at the Faculty Club of the University of California, Berkeley from 11–2 on Sunday, March 7. This event will include lunch (free of charge), a mathematical talk by Professor Alan Weinstein of UC Berkeley, and the awarding of over 60 prizes, worth approximately $5000 in total. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.