7th Bay Area Mathematical Olympiad

February 22, 2005

Problems and Solutions

1 An integer is called *formidable* if it can be written as a sum of distinct powers of 4, and *successful* if it can be written as a sum of distinct powers of 6. Can 2005 be written as a sum of a formidable number and a successful number? Prove your answer.

Solution: Suppose that $2005 = a + b$, where a is formidable and b is successful. Then a must be a sum of some of the powers of 4 less than 2005, namely 1, 4, 16, 64, 256, 1024. Similarly, b must be a sum of some of the numbers 1, 6, 36, 216, 1296. So, $a + b$ must be a sum of some distinct entries from the list

$$1, 1, 4, 6, 16, 36, 64, 216, 256, 1024, 1296.$$

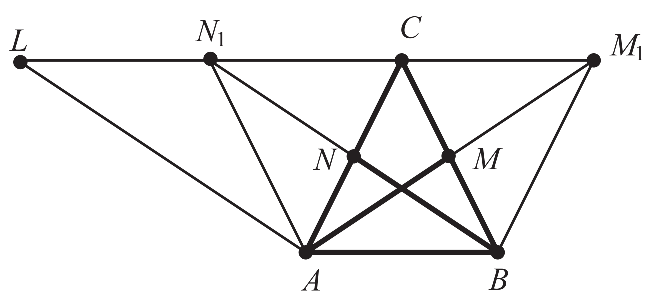
If we use both 1024 and 1296, we get at least $1024 + 1296 = 2320$ which is too big. But if we omit one of them, the most we can get is

$$1 + 1 + 4 + 6 + 16 + 36 + 64 + 216 + 256 + 1296 = 1896$$

which is too small. So there is no way to achieve a sum of 2005. ■

NOTE: We apologize for the ambiguity of the question, because we did not carefully define what was meant by a "power." Some students assumed that powers had to be positive, which meant formidable and successful numbers are always even, and consequently 2005 cannot be a sum of a formidable and a successful number. Other students assumed that negative powers were possible (which did not alter the solution above in any important ways). Still others assumed that fractional powers were allowable. In this case, it *is* possible to express 2005 as a sum of a formidable number and a successful number. All of these "alternative" solutions, if carefully done, got full credit.

2 Prove that if two medians in a triangle are equal in length, then the triangle is isosceles.



Solution 1: Let equal medians AD and BE in triangle ABC meet at F. It is well known that

$$AF : FD = BF : FE = 2 : 1.$$

Triangle ABC is isosceles if we can show $AC = BC$ or equivalently, that $AE = BD$. But triangles AFE and BFD are congruent with vertical angles plus sides that are $1/3$ and $2/3$ of the equal median length. ■

Solution 2: Let medians $AM = BN$ in $\triangle ABC$. Extend each median to AM_1 and BN_1 so that M and N are the midpoints of AM_1 and BN_1, respectively. By the property of bisecting diagonals, ABM_1C and $ABCN_1$ are parallelograms. Hence CM_1 and CN_1 are each parallel and equal to AB. We conclude that C lies on N_1M_1, C is the midpoint of N_1M_1, and $AM_1 = BN_1$ as they are twice the lengths of the original medians AM and BN. Summarizing, ABM_1N_1 is a trapezoid with equal diagonals.

It is easy to see that such a trapezoid is isosceles. One way to see this is to draw a line through A parallel to diagonal BN_1, until it intersects line N_1M_1 in point L. Thus, ABN_1L is a parallelogram, so $\angle ALN_1 = \angle ABN_1$. On the other hand, $\triangle AM_1L$ is isosceles since $AL = BN_1 = AM_1$; hence, $\angle ALN_1 = \angle AM_1N_1$. Finally, $AB||N_1M_1$ implies $\angle AM_1N_1 = \angle BAM_1$. We conclude that $\angle BAM_1 = \angle ABN_1$, and $\triangle ABN_1$ and $\triangle BAM_1$ are congruent by two equal sides and angles between these sides. Therefore, $BM_1 = AN_1$ and our trapezoid is isosceles. Hence $\angle AN_1C = \angle BM_1C$.

Finally, $\triangle ACN_1$ and $\triangle BCM_1$ are congruent by $AN_1 = BM_1$, $CN_1 = CM_1$ and $\angle AN_1C = \angle BM_1C$. We conclude that $AC = BC$ and our original $\triangle ABC$ is also isosceles. ■

Solution 3: As a variation of the above solution, note that NM is the midsegment of $\triangle ABC$, and as such it is parallel to AB. Thus $ABMN$ is a trapezoid with equal diagonals, which by a similar argument as in Solution 1 is isosceles. Therefore, $\angle BAC = \angle ABC$ and $AC = BC$. ■

Solution 4: A well-known formula for a parallelogram ABM_1C says: $2(AB^2 + AC^2) = AM_1^2 + BC^2$ (it can be easily proved with vectors for example). From here one derives a formula for the median AM of a triangle $\triangle ABC$:

$$AM^2 = \frac{1}{2}(AB^2 + AC^2) - \frac{1}{4}BC^2.$$

Similarly, the other median BN in $\triangle ABC$ satisfies:

$$BN^2 = \frac{1}{2}(AB^2 + BC^2) - \frac{1}{4}AC^2.$$

Since $AM = BN$, easy algebraic cancellations lead to $AC^2 = BC^2$, i.e. $AC = BC$ and our triangle is isosceles. ■

3 Let n be an integer greater than 12. Points P_1, P_2, $\ldots$, P_n, Q in the plane are distinct. Prove that for some i, at least $n/6 - 1$ of the distances

$$P_1P_i,\ P_2P_i,\ \ldots,\ P_{i-1}P_i,\ P_{i+1}P_i,\ \ldots,\ P_nP_i$$

are less than P_iQ.

Solution: Cut the plane into six 60° "pizza slices" with vertex Q. Rotating if necessary, we may assume that none of the P_j lie on the cuts. By the pigeonhole principle, one slice contains at least $n/6$ of the P_j. Let P_i be a point in this slice farthest from Q. It remains to show that all other points P_j in this slice satisfy $P_j P_i < P_i Q$.

The average of the angles of $\Delta P_j P_i Q$ is $180^\circ/3 = 60^\circ$, so $\angle P_j Q P_i$, which is less than 60°, is less than one of the other angles. Smaller angles of a triangle are opposite shorter sides, so $P_j P_i$ is less than one of $P_j Q$ and $P_i Q$. By choice of i, $P_j Q \le P_i Q$, so in any case, $P_j P_i < P_i Q$. (Alternatively, one could use the Law of Cosines to show that the side of a triangle opposite an angle smaller than 60° is not the longest.) ■

4 There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

Solution 1: Call a city "even" or "odd" according to whether the number of paved roads coming out of it is even or odd. Note the following.

> **Lemma:** *No matter which roads are paved, there will be an even (possibly zero) number of even cities.*

To see why this is true, let d_i be the number of paved roads leading out of city i. The sum

$$d_1 + d_2 + \cdots + d_{1000}$$

counts each paved road exactly twice, and hence will be an even number (this is known as the "Handshake Lemma"). If there were an odd number of even cities, then there would be an odd number of odd cities (since there are 1000 cities and 1000 is even). But then the sum above would be odd, a contradiction.

Using this lemma, we can create an algorithm which will eventually pave the roads so that all cities are odd.

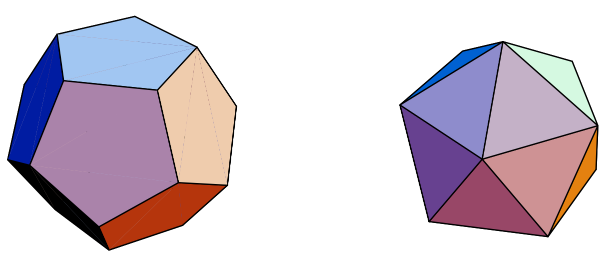
1. Start by paving all the roads. If each city is odd, we are done.
2. Otherwise, find an even city x. By the lemma, there will be at least one other even city, y. Consider the path joining x and y. Change the "state" of all the roads in this path (in other words, if a road is paved, unpave it; if it is dirt, pave it). This procedure changes the parity of x and y (i.e., changes them from even to odd), but does not alter the parity of any other city in Euleria, because if z is a city on the path from x to y, both the road going into z (from the x-direction) and the road leaving it (heading towards y) will have changed.
3. Step 2 thus reduces the number of even cities by 2. Repeat this step as much as needed until the number of even cities is zero. ■

Solution 2: We will think of Euleria as a connected graph (a network of vertices joined by edges), where each city is a vertex and each dirt road joining two cities is an edge. Number the vertices $1, 2, \ldots, 1000$. For each $i = 1, 2, \ldots, 500$, consider any path from vertex $2i - 1$ to vertex $2i$ (we know such a path exists, since the graph is connected), and place a mark on each edge used in the path. Then pave all the roads (edges) that have an odd number of marks. To show that this scheme meets the requirements, it is enough to prove that, for each vertex v, the edges incident to it have an odd total number of marks, since the number of such edges with an odd number of marks will then be odd. But consider all the occurrences of v in any of our 500 paths. Notice that v is an endpoint of one such path, which therefore contributes one mark on an edge incident to v; any other occurrence of v is internal to a path, which therefore contributes two marks, one on the edge leading into v and another on the edge leading out of it. It follows that the total number of marks on edges incident to v is odd. This completes the proof. ■

5 Let D be a dodecahedron which can be inscribed in a sphere with radius R. Let I be an icosahedron which can also be inscribed in a sphere of radius R. Which has the greater volume, and why?

Note: A *regular polyhedron* is a geometric solid, all of whose faces are congruent regular polygons, in which the same number of polygons meet at each vertex. A regular *dodecahedron* is a polyhedron with 12 faces which are regular pentagons and a regular *icosahedron* is a polyhedron with 20 faces which are equilateral triangles. A polyhedron is *inscribed in a sphere* if all of its vertices lie on the surface of the sphere.

The illustration below shows a dodecahdron and an icosahedron, not necessarily to scale.



Solution 1: D has the greater volume.

The key idea that we will use is a beautiful relationship between I and D; they are *dual* polyhedra. To understand this, note that D has 12 faces, 30 edges, and 20 vertices. (The 12 pentagonal faces produce 60 edges and 60 vertices, but these are, respectively, double- and triple-counted. Hence there are $60/2 = 30$ edges and $60/3 = 20$ vertices). Likewise, I has 20 faces, 30 edges, and 12 vertices. Notice that I has as many vertices as D has faces, and vice-versa.

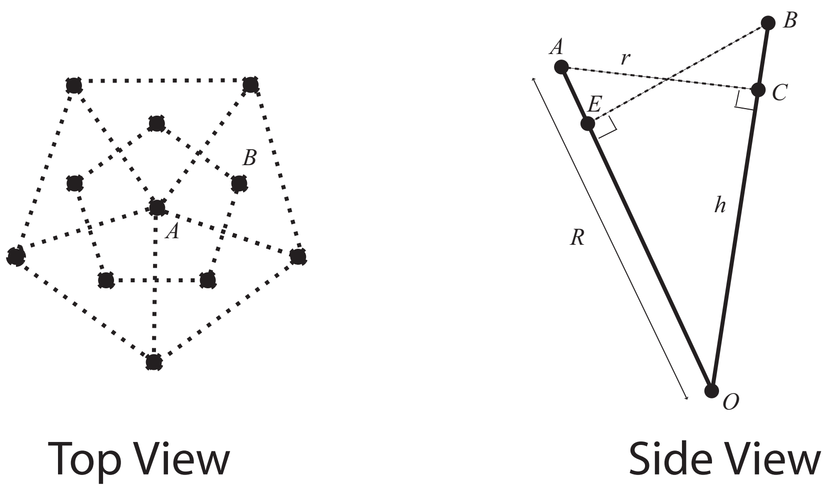
Consequently, if we join each center of the faces of an icosahedron to the centers of adjacent faces, we will get a dodecahedron, and vice-versa. (For the other platonic solids, the cube and octahedron are duals of each other, and the tetrahedron is its own dual).

We will use duality to prove an important lemma.

Lemma: *If a dodecahedron and an icosahedron can be inscribed in the same sphere, then they circumscribe the same sphere as well.*

To see why this is true, imagine I and D both sitting inside a sphere of radius R with center O, which circumscribes both polyhedra. Thus all the vertices of both polyhedra lie on the surface of this sphere. *Because the polyhedra are dual*, we can place the polyhedra so that for every vertex A of I, the line AO is perpendicular to and passes through the center of a face of D. Likewise, for every vertex B of D, the line BO is perpendicular to and passes through the center of a face of I.

The illustration below is an imperfect depiction of a portion of this situation. The left picture is a "top view," as seen by an observer looking directly down at the center of a face of D. Point A is a vertex of I, and B is a vertex of D.



The picture on the right is a "side view" (not-to-scale), showing the projections of A and B down to the center O of the sphere. Let CO be the perpendicular projection of AO onto BO. Note that point C is the center of a face of the icosahedron I (in the Top View, you cannot see C because it is directly "underneath" B), and $h = CO$ is the radius of the *inscribed* sphere of I. Also $r = AC$ is the radius of the circumscribed circle about a *face* of I.

Let EO be the perpendicular projection of BO on to AO. Clearly $EO = CO = h$ (since $AO = BO = R$), but by duality, E must be a center of a face of the dodecahedron D. Consequently, the radius of the inscribed sphere of D must also equal h.

Now it is a simple matter to compute and compare the two volumes. Notice that D consists of 12 pyramids with regular pentagonal bases (each with circumradius r) and height h. Likewise, I consists of 20 pyramids with equilateral triangular bases (each with circumradius r) and height h. Recall that the volume of a pyramid with base area B and height h is $Bh/3$. Now we can easily compare the two volumes.[1]

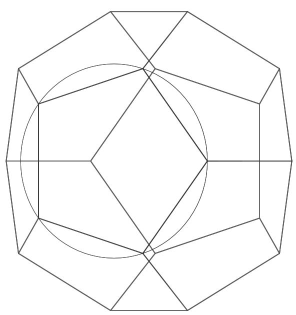
$$\frac{\text{vol}(D)}{\text{vol}(I)} = \frac{\dfrac{h}{3} \times 12 \times \text{ area of pentagon}}{\dfrac{h}{3} \times 20 \times \text{ area of triangle}}$$

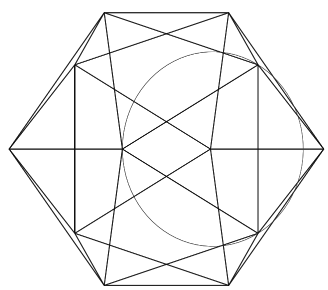
[1]We are using the fact that a regular n-gon that can be inscribed in a circle of radius r can be dissected into n "pie slices" that are isosceles triangles with sides of length r and vertex angle $(360/n)^\circ$. Thus the area of the regular n-gon is $n \times (r^2/2) \times \sin(360/n)^\circ$.

$$
\begin{aligned}
&= \frac{3 \times \text{ area of pentagon}}{5 \times \text{ area of triangle}} \\
&= \frac{3 \times 5 \times \frac{r^2}{2} \sin 72^\circ}{5 \times 3 \times \frac{r^2}{2} \sin 120^\circ} \\
&= \frac{\sin 72^\circ}{\sin 60^\circ} \\
&> 1.
\end{aligned}
$$

■

Solution 2: (sketch) Instead of showing that the inscribed spheres have equal radii, we can use duality and the "top view" used in the previous solution to show that the radius of the circumscribed circle of each face of D is equal to the radius of the circumscribed circle of each face of I. In other words, the two circles depicted below have the same radii. This fact is attributed to the ancient Greek mathematician Apollonius.





Once this is known, it is easy to deduce that the two polyhedra have identical inscribed sphere radii, and the rest of the solution proceeds as before.