10th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 26, 2008

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

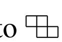
Note that the five problems are numbered 3–7. This is because BAMO-8, the middle-school version, has four problems, numbered from 1 to 4. The two hardest problems of BAMO-8 are the first two problems of BAMO-12. So collectively, the problems of the two BAMO exams are numbered 1–7.

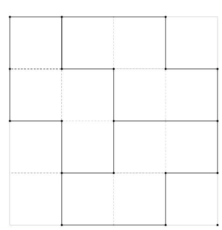
Problems

3 A triangle (with non-zero area) is constructed with the lengths of the sides chosen from the set

$$\{2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}.$$

Show that this triangle must be isosceles. (A triangle is *isosceles* if it has at least two sides the same length.)

4 Determine the greatest number of figures congruent to  that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares. The figures can be rotated and flipped over. For example, the picture below shows that at least 3 such figures can be placed in a 4×4 grid.



For this problem, please feel free to use the graph paper supplied by your proctor.

Please turn over for problems 5, 6, and 7.

5 N teams participated in a national basketball championship in which every two teams played exactly one game. Of the N teams, 251 are from California. It turned out that a California team, Alcatraz, is the unique California champion (Alcatraz won more games against California teams than any other team from California). However, Alcatraz ended up being the unique loser of the tournament because it lost more games than any other team in the nation!

What is the smallest possible value for N?

6 Point D lies inside the triangle ABC. Let A_1, B_1, and C_1 be the second intersection points of the lines AD, BD, and CD with the circles circumscribed about $\triangle BDC$, $\triangle CDA$, and $\triangle ADB$, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

7 A positive integer N is called *stable* if it is possible to split the set of all positive divisors of N (including 1 and N) into two subsets that have no elements in common, which have the same sum. For example, 6 is stable, because $1+2+3=6$, but 10 is not stable. Is $2^{2008} \cdot 2008$ stable?

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2008 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 9. This event will include lunch, a mathematical talk by John Conway of Princeton University, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.