10th Bay Area Mathematical Olympiad

February 26, 2008

Problems (with Solutions)

1 Call a year *ultra-even* if all of its digits are even. Thus $2000, 2002, 2004, 2006$, and 2008 are all ultra-even years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an ultra-even year because of the 9, and 2010 is not an ultra-even year because of the 1.

(a) In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them. Justify your answer.

(b) What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two ultra-even years? Again, give an example, and justify your answer.

Solution:

(a) The longest possible gap has length 1112. There are four valid examples of two ultra-even years that far apart with no ultra-even years between them: 888 to 2000, 2888 to 4000, 4888 to 6000 or 6888 to 8000. (8888 to 10000 is *not* acceptable as 10000 is not ultra-even).

To see that no *longer* gap is possible, note that if A and B are ulta-even years between 1 and 10000 with no ulta-even year between them, and $B - A \geq 1000$, then A and B must have different digits in their ”thousands place”. Since both of these digits must be even, that means there must be at least one odd digit d in between them, and so all the years between $d000$ and $d999$ must be in the gap between A and B. But the smallest ultra-even year greater than $d999$ is $e000$ (where $e = d + 1$, of course), and the largest ultra-even year less than $d000$ is $c888$ (where $c = d - 1$), so this gap must be of length exactly 1112.

Another approach is to consider the increasing sequence of ulta-even years: 2 (the smallest ultra-even year), 888, 2000, 2888, 4000, 6000, 6888, 8000, 8888 (the largest ultra-even year less than 10000). Every ultra-even year is either in this sequence or lies between two elements of this sequence, so any two ultra-even years with no ultra-even years between them must have a difference that is less than or equal to a difference between two consecutive elements of this set; but the largest such difference is 1112. ∎

(b) The second-shortest possible gap is 12. Examples include 8 to 20, 4268 to 4280, etc

To see that this really is the second-shortest possible gap, note that every gap must be even, since the difference between any two ultra-even years is even. There are certainly many gaps of length 2. (from 4022 to 4024, for example). We must show that there are no gaps with lengths 4,6,8, or 10. Any ultra-even year whose UNITS digit is 0,2,4, or 6 is 2 less than another ulta-even year (since adding 2 to this year won't change any other digit). So if an ultra-even year is the lower year of a gap longer than 2 years, its units digit must be 8. However, if year A is ultra-even, and has a units digit of 8, adding 2,4,6, 8, or 10 to it will result in a year whose ten digit is one greater than the ten digit of year A, which will mean the new number is not ultra-even. So 12 is the smallest possible value for the gap between such years A and the next largest ultra-even number. ∎

2 Consider a 7×7 chessboard that starts out with all the squares white. We start painting squares black, one at a time, according to the rule that after painting the first square, each newly painted square must be adjacent along a side to only the square just previously painted. The final figure painted will be a connected “snake” of squares.

(a) Show that it is possible to paint 31 squares.

(b) Show that it is possible to paint 32 squares.

(c) Show that it is possible to paint 33 squares.

Solution: (a) (b) (c)

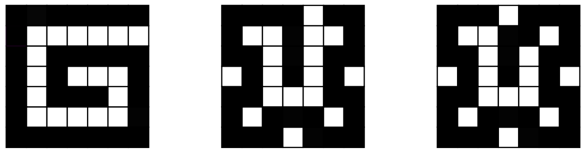


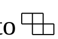
Figure 1: Solutions to (a), (b) and (c)

3 A triangle (with non-zero area) is constructed with the lengths of the sides chosen from the set

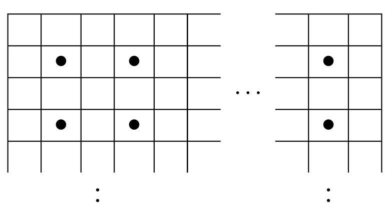
$$\{2,3,5,8,13,21,34,55,89,144\}.$$

Show that this triangle must be isosceles (A triangle is *isosceles* if it has at least two sides the same length.)

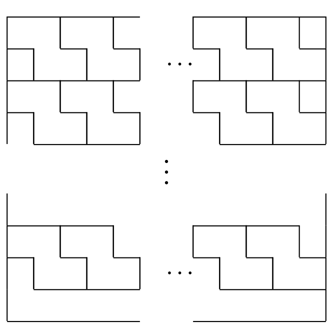
Solution: First note that the numbers in the list are an increasing sequence of Fibonacci numbers, i.e. starting with 5, each number is the sum of the previous two numbers in the list. By the triangle inequality, we know that the longest side is always less than the sum of the other two sides. Suppose the triangle constructed is scalene with sides $a < b < c$. If a and b are consecutive numbers in the list then the longest side c is the next number in the list or a larger one and $a+b \le c$, a contradiction of the triangle inequality. If a and b are not consecutive numbers in the list, let d be the number just before b, so that $a < d < b$. We then have $a+b < d+b$. But by the previous argument $d+b \le c$, so by the transitivity of '$<$' we have $a+b < c$, again contradicting the triangle inequality. Therefore the supposition is false and the triangle cannot be scalene and must have at least two sides equal. ■

4 Determine the greatest number of figures congruent to  that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares.

Solution: Consider the marked squares in the following picture.



Every figure covers exactly one marked square. Since the number of marked squares is equal to $\left\lfloor \frac{9}{2} \right\rfloor \cdot \left\lfloor \frac{9}{2} \right\rfloor = 16$, it is possible to place at most 16 figures. It is indeed possible to place that many figures as shown in the following picture:



5 N teams participated in a national basketball championship in which every two teams played exactly one game. Of the N teams, 251 are from California. It turned out that a Californian team Alcatraz is the unique Californian champion (Alcatraz has won more games against Californian teams than any other team from California). However, Alcatraz ended up being the unique loser of the tournament because it lost more games than any other team in the nation!

What is the smallest possible value for N?

Solution: We will prove that $N = 255$ is the smallest value.

- Let us first construct a tournament with the described properties and 255 participating teams. First arrange 251 Californian teams in the circle and label them by 0, 1, ..., 250 in the counter-clockwise direction (0 is Alcatraz). If each team won the games against its first 125 opponents in the counter-clockwise direction and lost against the other opponents, then each team has won exactly 125 games. However, if we look at the tournament with all outcomes the same except for Alcatraz winning the game against the team 250 (instead of losing it), then Alcatraz is the unique Californian champion, and 250 is the unique Californian loser. From now on, let us denote by L that unique Californian loser. L has won 124 and Alcatraz has won 126 games. There remain 249 teams in California, each of which won exactly 125 games. Let us split them into two sets P and Q containing 125 and 124 teams, respectively.

  Now we will add the remaining 4 non-Californian teams. Denote them by A, B, C, D. They should all win against Alcatraz (then Alcatraz has exactly 126 wins), A and B should beat all teams in P and lose against teams in Q. C and D should do exactly the opposite. Now each member of P and Q has 127 wins, each of A, B has 126, while C and D won 125 times. Let us make A win against L, then A won 127 times; let us make L win against B, C, D. Then L has also 127 victories. If B wins the game against A, then it will have 127 victories as well. If C and D win against A and B they will have 127 victories. Now each team except for Alcatraz has 127 victories. (There is still one game remaining—the one between C and D, we don't care about that one.)

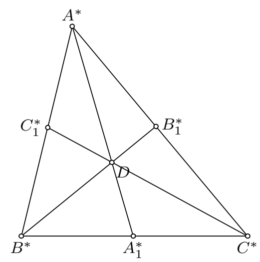
- Now we will prove that there is no tournament with less than 4 foreign teams. First of all, Alcatraz had to have at least 126 wins; otherwise there will be at most $124 \cdot 250 + 1$ victories in the Californian sub-tournament, but the total number of games (and hence victories) is $125 \cdot 251$. Other teams in the tournament had to win at least 127 times. However, in the Californian sub-tournament, there is a team who won in no more than 124 games (otherwise the number of wins would be $\geq 125 \cdot 250 + 126 > 250 \cdot 251/2$ – a contradiction). Denote one such team by L. We immediately conlude that there are at least 3 foreign teams who lost to L. Assume that there are only three non-California teams. Except for Alcatraz, each of the Californian teams won in at least two of the games against foreigners which amounts to not less than $250 \cdot 2$ Californian victories. There are $3 \cdot 251$ such matches, so non-Californians could win in at most 253 games against Californians. They played additional 3 games among themselves, so they made at most 256 victories, which is a contradiction to the fact that each of them won at least 127 times. Therefore $N \geq 251 + 4 = 255$.

Note that it is crucial to do both parts of this argument: show that 255 is minimal, *and* construct an actual tournament to show that the required conditions actually happen. ■

6 Point D lies inside the triangle ABC. If A_1, B_1, and C_1 are the second intersection points of the lines AD, BD, and CD with the circles circumscribed about $\triangle BDC$, $\triangle CDA$, and $\triangle ADB$, prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

Solution: Let k be the circle with center D and radius 1. Consider the inversion with respect to the circle k and denote by A^*, B^*, C^*, A_1^*, B_1^*, and C_1^* the images of A, B, C, A_1, B_1, and C_1, respectively.



The point C_1^* belongs to the line A^*B^*, because the circumcircle of $\triangle ADB$ is mapped to a line. Simiarly, $B_1^* \in B^*C^*$ and $A_1^* \in A^*B^*$. D belongs to the line $A^*A_1^*$ and $AD = \frac{1}{A^*D}$. Using the similar reasoning we get that D is the intersection of $A^*A_1^*$, $B^*B_1^*$, and $C^*C_1^*$ and the desired equality now becomes equivalent to

$$\frac{A_1^*D}{A^*A_1^*} + \frac{B_1^*D}{B^*B_1^*} + \frac{C_1^*D}{C^*C_1^*} = 1. \tag{0.1}$$

Notice that $\frac{A_1^*D}{A^*A_1^*} = \frac{S_{\triangle B^*C^*D}}{S_{\triangle B^*C^*A^*}}$. Analogous relations for the remaining two fractions on the left-hand side of (**??**) further transform our claim to:

$$\frac{S_{\triangle B^*C^*D}}{S_{\triangle A^*B^*C^*}} + \frac{S_{\triangle C^*A^*D}}{S_{\triangle A^*B^*C^*}} + \frac{S_{\triangle A^*B^*D}}{S_{\triangle A^*B^*C^*}} = 1,$$

which is obviously true. ■

7 A positive integer N is called *stable* if it is possible to split the set of all positive divisors of N (including 1 and N) into two subsets that have no elements in common, which have the same sum. For example, 6 is stable, because $1+2+3=6$ but 10 is not stable. Is $2^{2008} \cdot 2008$ stable?

Solution 1: Yes. In general, let N be a number of the form $N = 2^k p$, where p is an odd prime less than 2^{k+1}. We will show that one can form an expression, obtained by adding and subtracting together all the divisors of N, which is equal to zero. First note that

$$2^k p - 2^{k-1} p - \cdots - p = p.$$

The remaining divisors are 1, 2, ..., 2^k, whose sum is more than p. Thus we are able to write $(2^{k+1} - 1 + p)/2$ as a sum of some subset of the remaining divisors simply by considering its binary representation. Clearly the unused terms will sum to $(2^{k+1} - 1 - p)/2$. But the difference between these two quantities is p, just as above. It is now clear how to form the desired expression involving all divisors of N which evaluates to zero, and you're done.

Solution 2: (Sketch) This solution, by David Spies of Albany High School, won the Brilliancy Award, because it used ideas that no other solution had.

If N is to be stable, then the factors of N can be put on either side of an equal sign, and when these factors are summed, you get an equality. The highest power of 2 that divides N will either lie on the same side as N, or the opposite side. The key idea of David's solution is the observation that if N is stable and the highest power of 2 that divides N lies on the opposite side of the equal sign from N, then $2N$ is also stable.

For example, 12 is stable, because

$$1+3+4+6 = 2+12,$$

and notice that 4 lies opposite 12. To show that 24 is stable, we let 4 and 12 exchange places, and then add in the two new factors (8 and 24), getting

$$1+3+12+6+8 = 4+24.$$

It is easy to see why this method works (verify it!)

The remainder of David's argument looks at 2008, which is not stable, but he successively doubles it, making it more and more balanced, until he shows that $2008 \cdot 2^4$ is stable. Then the algorithm described above will show that $2008 \cdot 2^k$ is stable for all $k \geq 4$. ■