13th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 22, 2011

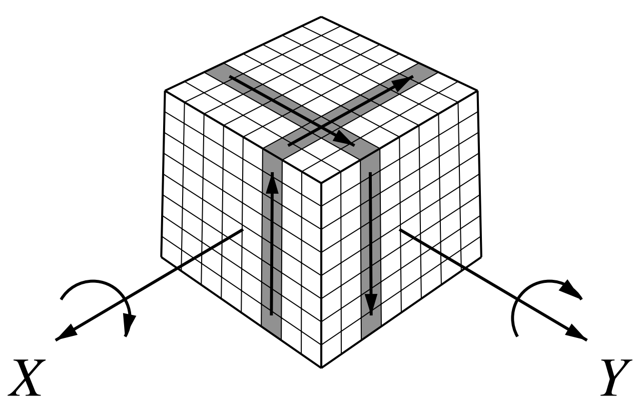
The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 Consider the $8 \times 8 \times 8$ Rubik's cube below. Each face is painted with a different color, and it is possible to turn any layer, as you can with smaller Rubik's cubes. Let X denote the move that turns the shaded layer shown (indicated by arrows going from the top to the right of the cube) clockwise by 90 degrees, about the axis labeled X. When move X is performed, the only layer that moves is the shaded layer. Likewise, define move Y to be a clockwise 90-degree turn about the axis labeled Y, of just the shaded layer shown (indicated by the arrows going from the front to the top, where the front is the side pierced by the X rotation axis). Let M denote the move "perform X, then perform Y."



Imagine that the cube starts out in "solved" form (so each face has just one color), and we start doing move M repeatedly. What is the least number of repeats of M in order for the cube to be restored to its original colors?

Please turn over for the remaining problems!

2 In a plane, we are given line l, two points A and B neither of which lies on line l, and the reflection A_1 of point A across line l. Using only a straightedge, construct the reflection B_1 of point B across line l. Prove that your construction works.

Note: "Using only a straightedge" means that you can perform only the following operations:

(a) Given two points, you can construct the line through them.

(b) Given two intersecting lines, you can construct their intersection point.

(c) You can select (mark) points in the plane that lie on or off objects already drawn in the plane. (The only facts you can use about these points are which lines they are on or not on.)

3 Let S be a finite, nonempty set of real numbers such that the distance between any two distinct points in S is an element of S. In other words, $|x-y|$ is in S whenever $x \neq y$ and x and y are both in S.

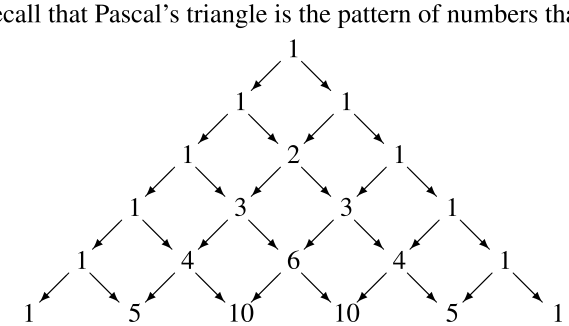
Prove that the elements of S may be arranged in an arithmetic progression. This means that there are numbers a and d such that $S = \{a, a+d, a+2d, a+3d, \ldots, a+kd, \ldots\}$.

4 Three circles k_1, k_2, and k_3 intersect in point O. Let A, B, and C be the second intersection points (other than O) of k_2 and k_3, k_1 and k_3, and k_1 and k_2, respectively. Assume that O lies inside of the triangle ABC. Let lines AO, BO, and CO intersect circles k_1, k_2, and k_3 for a second time at points A', B', and C', respectively. If $|XY|$ denotes the length of segment XY, prove that

$$\frac{|AO|}{|AA'|} + \frac{|BO|}{|BB'|} + \frac{|CO|}{|CC'|} = 1.$$

5 Does there exist a row of Pascal's Triangle containing four distinct values a, b, c and d such that $b = 2a$ and $d = 2c$?

Recall that Pascal's triangle is the pattern of numbers that begins as follows



where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4+6$.

Also note that the last row displayed above contains the four elements $a = 5, b = 10, d = 10, c = 5$, satisfying $b = 2a$ and $d = 2c$, but these four values are NOT distinct.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2011 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 13. This event will include lunch, a mathematical talk, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.