14th Bay Area Mathematical Olympiad and 3rd Teachers' Olympiad

February 26, 2012

Problems

1 In January, you could buy a bag of 40 apples or a bag of 60 oranges for one dollar. In February, they were sold as a single bag of 25 apples and 25 oranges for 1 dollar a bag.

Note that the average of 40 and 60 in January's situation is the same as the sum of 25 and 25 in February's situation.

a) Write and solve a word problem using these data in which the fact of the equal averages leads to equal answers.
b) Write and solve a word problem using these data in which the answers are unequal in spite of the equal averages.
c) Explain the vital characteristic(s) that distinguish these two types of questions.

2 Beginning with two 1s, at each step you insert into the gap between each pair of adjacent numbers, the sum of those two adjacent numbers. So after step 1 you have 1 2 1, and after step 2 you have 1 3 2 3 1, and so on. After step 2012, how many 2012s are there?

3 A previous teacher left you with a great collection of lesson plans. One day's lesson begins with

> A $3-4-5$ right triangle has a perimeter of 12 units and an area of 6 square units, so we can plot the point $(12,6)$ to represent this triangle.

Unfortunately, the rest of the lesson plan was lost. What topic(s) might be taught in the rest of this lesson, and what questions would you pose to lead students toward those ideas?

4 Explain the mistakes in the student solution below.

Problem: A standard deck of 52 cards, containing 4 aces and 4 kings, is shuffled and the cards are then arranged in a large circle. What is the probability that no aces are next to kings?

Solution: There are 52! ways to arrange the deck.

Out of these, there are $\binom{52}{4}$ ways for the kings to be located. Each king eliminates the two adjacent places, so the four kings leave $52-4\cdot 3=40$ spaces for the aces. Thus there are $\binom{40}{4}$ ways to arrange the aces. Our final probability is

$$\frac{\binom{52}{4}\cdot\binom{40}{4}}{52!}.$$

5 A student has recently learned that for any two positive integers x and y, the product of their greatest common divisor, often written (x,y), and their least common multiple, often written $[x,y]$, is equal to the product of the two numbers. That is, $(x,y)\cdot[x,y]=x\cdot y$.

This same student claims that you can use the same idea for three numbers: $(x,y,z)\cdot[x,y,z]=x\cdot y\cdot z$. Are there examples in which this student's idea would be true? How could you fix the student's formula to give correct answers for all positive integers x,y,z?

Turn over the page to find the remaining problems!

6 A student who recently learned the distributive property in algebra class decides to "distribute" $a+(b\cdot c)$ as $(a+b)\cdot(a+c)$.

How would you explain why this procedure is not correct? Give several different explanations.

What are the differences between this and the usual $a\cdot(b+c)=a\cdot b+a\cdot c$?

Are there any circumstances in which this sort of "distributive property" actually gives correct answers?

7 In an algebra 1 textbook, you find the following problem:

> One hose can fill your swimming pool in 14 hours. With a second hose, working together, the two take only 6 hours. How long does the second hose take when working by itself?

(a) Solve the problem, explaining your work in such a way that an algebra 1 student would understand it clearly.

(b) You want to write a version of this problem for students to practice with. What number(s) can you replace the 14 with in order to get integer answers to this question?

(c) On an exam, you want to have several different versions of approximately equal difficulty. What number(s) can you replace the 6 with such that there are at least six different possible numbers in place of the 14 that lead to integer answers?

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2012 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 11. This event will include lunch, a mathematical talk by Professor Jim Propp, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.