15th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 26, 2013

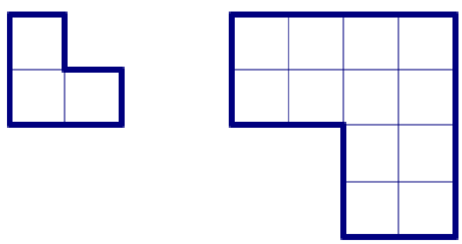
The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

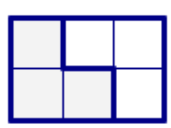
The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 Define a *size-n tromino* to be the shape you get when you remove one quadrant from a $2n \times 2n$ square. In the figure below, a size-1 tromino is on the left and a size-2 tromino is on the right.



We say that a shape can be *tiled with size-1 trominos* if we can cover the entire area of the shape—*and no excess area*—with *non-overlapping* size-1 trominos. For example, a 2×3 rectangle can be tiled with size-1 trominos as shown below, but a 3×3 square cannot be tiled with size-1 trominos.



a) Can a size-5 tromino be tiled by size-1 trominos?

b) Can a size-2013 tromino be tiled by size-1 trominos?

Justify your answers.

Please turn over for the remaining problems!

2 For a positive integer $n > 2$, consider the $n-1$ fractions

$$\frac{2}{1}, \frac{3}{2}, \dots, \frac{n}{n-1}.$$

The product of these fractions equals n, but if you reciprocate (i.e. turn upside down) some of the fractions, the product will change. Can you make the product equal 1? Find all values of n for which this is possible and prove that you have found them all.

3 Let H be the orthocenter of an acute triangle ABC. (The *orthocenter* is the point at the intersection of the three altitudes. An *acute triangle* has all angles less than $90°$.) Draw three circles: one passing through A, B, and H, another passing through B, C, and H, and finally, one passing through C, A, and H. Prove that the triangle whose vertices are the centers of those three circles is congruent to triangle ABC.

4 Consider a rectangular array of single digits $d_{i,j}$ with 10 rows and 7 columns, such that $d_{i+1,j} - d_{i,j}$ is always 1 or -9 for all $1 \le i \le 9$ and all $1 \le j \le 7$, as in the example below. For $1 \le i \le 10$, let m_i be the median of $d_{i,1}, \dots, d_{i,7}$. Determine the least and greatest possible values of the mean of $m_1, m_2, \dots, m_{10}$.

Example:

	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$d_{i,4}$	$d_{i,5}$	$d_{i,6}$	$d_{i,7}$	m_i
$i=1$	2	7	5	9	5	8	6	median is 6
$i=2$	3	8	6	0	6	9	7	median is 6
$i=3$	4	9	7	1	7	0	8	median is 7
$i=4$	5	0	8	2	8	1	9	median is 5
$i=5$	6	1	9	3	9	2	0	median is 3
$i=6$	7	2	0	4	0	3	1	median is 2
$i=7$	8	3	1	5	1	4	2	median is 3
$i=8$	9	4	2	6	2	5	3	median is 4
$i=9$	0	5	3	7	3	6	4	median is 4
$i=10$	1	6	4	8	4	7	5	median is 5

5 Let $F_1, F_2, F_3 \dots$ be the *Fibonacci sequence*, the sequence of positive integers with $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 1$. A *Fibonacci number* is by definition a number appearing in this sequence.

Let $P_1, P_2, P_3, \dots$ be the sequence consisting of all the integers that are products of two Fibonacci numbers (not necessarily distinct) in increasing order. The first few terms are

$$1, 2, 3, 4, 5, 6, 8, 9, 10, 13, \dots$$

since, for example $3 = 1 \cdot 3, 4 = 2 \cdot 2$, and $10 = 2 \cdot 5$.

Consider the sequence D_n of *successive differences* of the P_n sequence, where $D_n = P_{n+1} - P_n$ for $n \ge 1$. The first few terms of D_n are

$$1, 1, 1, 1, 1, 2, 1, 1, 3, \dots.$$

Prove that every number in D_n is a Fibonacci number.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2013 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 10. This event will include lunch, a mathematical talk, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.