17th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 24, 2015

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 Which number is larger, A or B, where

$$A = \frac{1}{2015}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2015}\right) \quad \text{and} \quad B = \frac{1}{2016}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2016}\right)?$$

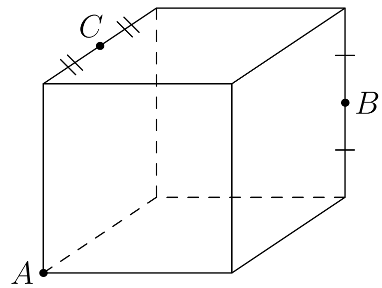
Prove that your answer is correct.

2 In a quadrilateral the two segments connecting the midpoints of its opposite sides are equal in length. Prove that the diagonals of the quadrilateral are perpendicular. (In other words, let M, N, P, and Q be the midpoints of sides AB, BC, CD, and DA in quadrilateral $ABCD$. It is known that segments MP and NQ are equal in length. Prove that AC and BD are perpendicular.)

3 Let k be a positive integer. Prove that there exist integers x and y, neither of which is divisible by 3, such that $x^2 + 2y^2 = 3^k$.

Please turn over for the remaining problems!

4 Let A be a corner of a cube. Let B and C be the midpoints of two edges in the positions shown on the figure below:



The intersection of the cube and the plane containing A, B, and C is some polygon, $\mathcal{P}$.

(a) How many sides does $\mathcal{P}$ have? Justify your answer.

(b) Find the ratio of the area of $\mathcal{P}$ to the area of $\triangle ABC$ and prove that your answer is correct.

5 We are given n identical cubes, each of size $1 \times 1 \times 1$. We arrange all of these n cubes to produce one or more congruent rectangular solids, and let $B(n)$ be the number of ways to do this. For example, if $n = 12$, then one arrangement is twelve $1 \times 1 \times 1$ cubes, another is one $3 \times 2 \times 2$ solid, another is three $2 \times 2 \times 1$ solids, another is three $4 \times 1 \times 1$ solids, etc. We do not consider, say, $2 \times 2 \times 1$ and $1 \times 2 \times 2$ to be different; these solids are congruent. You may wish to verify, for example, that $B(12) = 11$.

Find, with proof, the integer m such that $10^m < B(2015^{100}) < 10^{m+1}$.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2015 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 15 (note that is a week later than previous years). This event will include lunch, a mathematical talk, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.

 For more information about the awards ceremony, or with any other questions about BAMO, please contact Ian Brown at ibrown@proofschool.org.