A: There are 7 boxes arranged in a row and numbered 1 through 7. You have a stack of 2015 cards, which you place one by one in the boxes. The first card is placed in box #1, the second in box #2, and so forth up to the seventh card which is placed in box #7. You then start working back in the other direction, placing the eighth card in box #6, the ninth in box #5, up to the thirteenth card being placed in box #1. The fourteenth card is then placed in box #2, and this continues until every card is distributed. What box will the last card be placed in?

Solution: The answer is **box #3**. Card #1 is placed into box #1, and this gets visited again by card #13. Hence, box #1 is visited every 12 times. Since $2004 = 167 \cdot 12$, we see that card #2005 will be placed into box #1. Now, counting "by hand," we see that card #2015 will go into box #3.

B: Members of a parliament participate in various committees. Each committee consists of at least 2 people, and it is known that every two committees have at least one member in common. Prove that it is possible to give each member a colored hat (hats are available in black, white or red) so that every committee contains at least two members with different hat colors.

Solution: Pick a committee C of smallest size and give one of its members a black hat and the rest of its members a white hat. Give everyone else who is not in this committee a red hat. Committee C contains two colors of hats (black and white) by choice. Any other committee with the exact same members as C also has black and white hats. Any committee whose membership is not identical to C's membership must contain a member that is not in C, since it is not a subset of C, which was chosen to be a smallest committee. Therefore, it must contain a red hat. But it also contains either a white or a black hat, since it shares a member with C by assumption.

C/1: Which number is larger, A or B, where

$$A = \frac{1}{2015}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2015}\right) \quad \text{and} \quad B = \frac{1}{2016}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2016}\right)?$$

Prove that your answer is correct.

Solution: We claim that:

$$A = \frac{1}{2015}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2015}\right) > B = \frac{1}{2016}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2016}\right).$$

To prove this, let $S = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2015}$. Then $A = \frac{1}{2015}S$ and $B = \frac{1}{2016}\left(S+\frac{1}{2016}\right)$.

Thus, our proposed inequality can be written as:

$$\frac{1}{2015}S \overset{?}{>} \frac{1}{2016}\left(S+\frac{1}{2016}\right).$$

After multiplying both sides by $2015 \cdot 2016$ to clear some of the denominators, the proposed inequality becomes equivalent to:

$$2016S \overset{?}{>} 2015S + \frac{2015}{2016},$$

which, after subtracting $2015S$ from both sides, is equivalent in turn to:

$$S \overset{?}{>} \frac{2015}{2016}.$$

But $S > 1$, so it follows that $S > \frac{2015}{2016}$, establishing the last inequality and thereby proving all of the previous inequalities. In particular, the proposed original inequality is correct:

$$A = \frac{1}{2015}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2015}\right) > B = \frac{1}{2016}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2016}\right). \quad \square$$

D/2: In a quadrilateral, the two segments connecting the midpoints of its opposite sides are equal in length. Prove that the diagonals of the quadrilateral are perpendicular. (In other words, let M, N, P, and Q be the midpoints of sides AB, BC, CD, and DA in quadrilateral $ABCD$. It is known that segments MP and NQ are equal in length. Prove that AC and BD are perpendicular.)

Solution: We will use a well-known theorem from geometry. A *midsegment* in a triangle is called a segment that the midpoints of two of its sides.

Theorem. The midsegment in a triangle connecting two sides in a triangle is parallel to the third side and half of its length. In other words, if K and L are the midpoints of sides XZ and YZ of $\triangle XYZ$, then the midsegment KL is parallel to side XY and KL is half as long as XY.

- As a consequence, the four midpoints M, N, P, and Q in $ABCD$ in our problem form a *parallelogram* $MNPQ$. Indeed, since QP is parallel to AC (as a midsegment in $\triangle ACD$), and MN is parallel to AC (as a midsegment in $\triangle ACB$), it follows that QP and MN are parallel. They are also half as long as AC and hence equal in length. This means that quadrilateral $MNPQ$ have parallel and equal in length opposite sides, and hence it is a parallelogram.
- From our problem we know that the diagonals QN and MP of this parallelogram $MNPQ$ are equal in length. This means that the parallelogram is actually a rectangle (another famous theorem from geometry). So now we know that $MNPQ$ is a rectangle, i.e., PN and PQ are perpendicular.
- As midsegments in $\triangle ACD$ and $\triangle DBC$, QP and PN are parallel correspondingly to AC and DB. This implies that AC and BD are perpendicular to each other, completing our proof.

□

3: Let k be a positive integer. Prove that there exist integers x and y, neither of which is divisible by 3, such that $x^2 + 2y^2 = 3^k$.

Solution: For the first several values of k it is straight-forward to find solutions x_k and y_k satisfying $x_k^2 + 2y_k^2 = 3^k$, leading to this table of solutions.

k	x_k	y_k
1	1	1
2	1	2
3	5	1
4	7	4
5	1	11

The key is to realize that negating any value of x_k or y_k also yields a solution, so we may recast the table as follows.

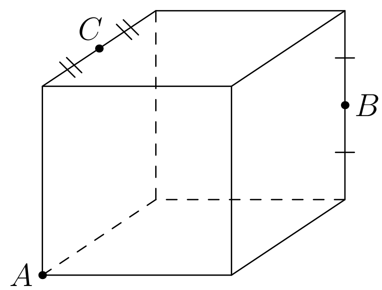
k	x_k	y_k
1	-1	-1
2	1	-2
3	5	-1
4	7	4
5	-1	11

It is now apparent that we should take $x_{k+1} = x_k - 2y_k$ and $y_{k+1} = x_k + y_k$. One then confirms that

$$\begin{aligned} x_{k+1}^2 + 2y_{k+1}^2 &= (x_k^2 - 4x_ky_k + 4y_k^2) + 2(x_k^2 + 2x_ky_k + y_k^2) \\ &= 3(x_k^2 + 2y_k^2), \end{aligned}$$

from which it easily follows by induction that $x_k^2 + 2y_k^2 = 3^k$ for all $k \geq 1$. Finally, one can also show by induction that $x_k \equiv y_k \equiv (-1)^k \bmod 3$, hence none of the x_k or y_k are divisible by 3.

4: Let A be a corner of a cube. Let B and C be the midpoints of two edges in the positions shown on the figure below:



The intersection of the cube and the plane containing A, B, and C is some polygon, $\mathcal{P}$.

(a) How many sides does $\mathcal{P}$ have? Justify your answer.

(b) Find the ratio of the area of $\mathcal{P}$ to the area of $\triangle ABC$ and prove that your answer is correct.

First solution. Orient the cube so that A is the front, left, bottom corner (as in the diagram from the problem).

Extend AB, AC to points B', C' such that B, C are the midpoints of $\overline{AB'}$ and $\overline{AC'}$, respectively. Now B' and C are both in the plane of the top face of the original cube; B, C' are both in the plane of the back face; and A, C are both in the plane of the left face. (Figure 1, below, makes this clear by means of some additional cubes.)

Each of the six planes forming the faces of the cube leaves a linear trace in the plane of $\triangle ABC$. Three of these lines (corresponding to the top, back, and left faces) are $B'C$, BC', and AC. The lines corresponding to the bottom, front, and right faces are parallel to the preceding three, and respectively pass through A, A, and B. Figure 2 shows these three pairs of parallel lines. The region lying between each pair of parallel lines is $\mathcal{P}$. It is a pentagon.

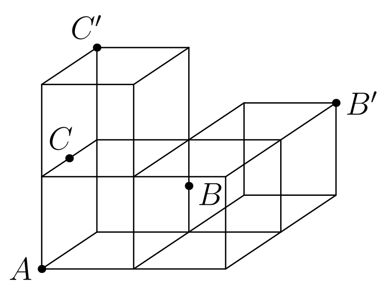


Figure 1

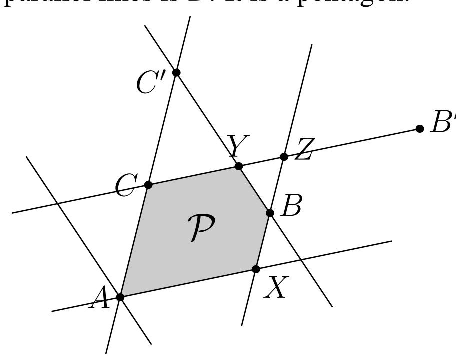


Figure 2

Let X, Y, Z be the intersections shown in Figure 2, noting that $AXZC$ is a parallelogram. Since B is halfway between the top and bottom faces of the cube, $BZ = \frac{1}{2} \cdot XZ$. Triangles $\triangle BZY$ and $\triangle C'CY$ are similar, and $C'C = XZ = 2 \cdot BZ$, so $CY = 2 \cdot ZY$, which implies that $ZY = \frac{1}{3} \cdot ZC$. Thus $\triangle BZY$ has one-third the base and one-half the height of parallelogram $AXZC$, so $[BZY] = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}[AXZC]$, making $[\mathcal{P}] = \frac{11}{12}[AXZC] = \frac{11}{12} \cdot 2[ABC]$. We conclude that the ratio of the area of $\mathcal{P}$ to the area of $\triangle ABC$ is $\boxed{11:6}$. ■

Second solution. Assign coordinates so as to place the corners of the cube at (x, y, z) with $x, y, z \in \{0, 1\}$, and so that $A = (0,0,0)$, $B = (1, 1, \frac{1}{2})$, and $C = (0, \frac{1}{2}, 1)$.

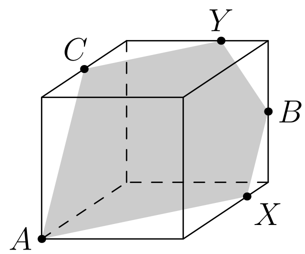
The equation of the plane containing A, B, C has the form $ax + by + cz = d$, where

$$a(0) + b(0) + c(0) = d,$$
$$a(1) + b(1) + c(\tfrac{1}{2}) = d,$$
$$a(0) + b(\tfrac{1}{2}) + c(1) = d.$$

A solution to this system is $a = 3$, $b = -4$, $c = 2$, $d = 0$. Thus the plane has equation

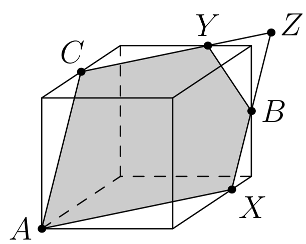
$$3x - 4y + 2z = 0.$$

Each vertex of $\mathcal{P}$ is on an edge of the cube and so has at least two coordinates which equal 0 or 1. There are $\binom{3}{2} \cdot 2^2 = 12$ ways to assign values of 0 or 1 to two coordinates. For each such assignment, we may substitute into the equation of the plane to obtain a point where the plane intersects a (possibly extended) edge of the cube. Some of these points are external to the cube. Checking all 12 cases, we obtain five distinct points that are on the cube and are therefore vertices of $\mathcal{P}$; those points are A, B, C, $X = (1, \frac{3}{4}, 0)$, and $Y = (\frac{2}{3}, 1, 1)$, shown below.



Thus $\mathcal{P}$ is a pentagon.

Now observe that $AX \parallel CY$ and $AC \parallel XB$. Thus we may extend XB and CY to meet at a point Z so that $AXZY$ is a parallelogram, which we call $\mathcal{P}'$:



Using the distance formula, we readily compute that $CY = \frac{2}{3} \cdot AX$ and $XB = \frac{1}{2} \cdot AC$. Thus $[\mathcal{P}] = [\mathcal{P}'] - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} [\mathcal{P}'] = \frac{11}{12} [\mathcal{P}'] = \frac{11}{12} \cdot 2[ABC] = \frac{11}{6} [ABC]$. That is, the ratio of the area of $\mathcal{P}$ to the area of $\triangle ABC$ is $\boxed{11 : 6}$. ■

5: We are given n identical cubes, each of size $1 \times 1 \times 1$. We arrange all of these n cubes to produce one or more congruent rectangular solids, and let $B(n)$ be the number of ways to do this. For example, if $n = 12$, then one arrangement is twelve $1 \times 1 \times 1$ cubes, another is one $3 \times 2 \times 2$ solid, another is three $2 \times 2 \times 1$ solids, another is three $4 \times 1 \times 1$ solids, etc. We do not consider, say, $2 \times 2 \times 1$ and $1 \times 2 \times 2$ to be different; these solids are congruent. You may wish to verify, for example, that $B(12) = 11$.

Find, with proof, the integer m such that $10^m < B(2015^{100}) < 10^{m+1}$.

Solution: The exact value of $B(2015^{100})$ is 921,882,251,894,177. Thus $m = 14$.

Let us estimate $B(n)$. First we note that the actual primes do not matter, just the exponents. Since $2015 = 5 \cdot 13 \cdot 31$, we need to find $B(p^{100}q^{100}r^{100})$, where p, q, r are distinct primes.

Let's first try an easier case: $n = p^3q^3r^3$. For each divisor of n, there will be one or more different "formations" of congruent rectangular solids. For example, we could take the divisor $d = p^2qr$ and one possible formation would be d solids, each with dimensions $p \times q \times qr^2$.

So each formation is a sort of 4-tuple, where the first coordinate is d, the number of solids, and the remaining three "coordinates" are the dimensions of the solid. In our example, the formation is the sort-of 4-tuple

$$(p^2qr; \{p, q, qr^2\}).$$

We call it a "sort-of" 4-tuple and use funny notation because the last three "coordinates" are an *unordered* trio, but the first coordinate—the number of solids—*matters*; it belongs in the first spot.

Making things even worse, *each* number $p^aq^br^c$ in our pseudo-4-tuple corresponds to an ordered triple (a, b, c) of exponents, where $0 \le a, b, c \le 3$. The example above is thus represented by

$$((2,1,1); \{(1,0,0), (0,1,0), (0,1,2)\}).$$

It is confusing, however, to combine ordered and non-ordered reasoning, so let us suppose, temporarily, that the order of the three dimension triples n the psudo-4-tuple *does* matter. In fact, let's assume that the order matters for all 4 triples. Then our pseudo-4-tuple (of ordered triples) becomes a genuine ordered 4-tuple of ordered triples. Thus the following 4-tuples are considered to be different:

$$((2,1,1); (1,0,0), (0,1,0), (0,1,2)),$$

$$((2,1,1); (0,1,0), (0,1,2), (1,0,0)),$$

$$((0,1,0); (2,1,1), (1,0,0), (0,1,2)).$$

Notice that the first two actually represent the same formation, but the last one is different.

This makes the counting much easier. For the divisor d with exponents (a, b, c), the possible dimensions are the three vectors

$$(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3),$$

where

$$\begin{aligned} a + a_1 + a_2 + a_3 &= 3, \\ b + b_1 + b_2 + b_3 &= 3, \\ c + c_1 + c_2 + c_3 &= 3 \end{aligned}$$

Each of these three equations represents a solution to a classic ball-and-urn counting problem with four distinguishable urns (since there are four terms) and three balls (since the sums are all equal to 3). For example, the number of *ordered* solutions to the first equation is the number of ordered 4-tuples (a, a_1, a_2, a_3) with coordinates adding to 3, and there are $\binom{6}{3}$ such 4-tuples.

Since we can pick solution 4-tuples for each of the three equations independently, there are $\binom{6}{3}^3$ different ordered solutions (ordered 4-tuples of ordered triples). For example, one solution may be

$$((2,1,1);(1,0,0),(0,1,0),(0,1,2)).$$

However, we are currently counting each of the 4! permutations of these as different. We want to order the count by the first triple (after all, this first triple indicates the number of blocks in the formation), but we do *not* want to count order among the other three triples.

Consequently, we just divide our current count by 3!, getting

$$\frac{\binom{6}{3}^3}{6}.$$

This is not an exact value (in fact, it is not even an integer), because it improperly accounts for solids where two or more of the dimensions are equal. For example, suppose that the number of solids is prq and the dimension of each of the the solids is $pr \times pr \times q^2$. Thus the first triple (for the number of solids) is $(1,1,1)$, and the three next triples (for the dimension) are $(1,0,1),(1,0,1)$, and $(0,2,0)$. These three triples do not have 6 different permutations, since two are equal; they have just 3 different permutations. Likewise, if the last three triples were the same (for example, if the dimensions of the solid was $pq \times pq \times pq$ then there is only one ordering of these three triples.

In the more general case, where $n = p^t q^t r^t$, our approximation would yield the formula $\binom{t+3}{3}^3/6$, which slightly misses the exact value. We say "slightly," because the the formula is a degree-9 polynomial whose first term is $t^9/1296$, but the number of formations for which there are two equal dimensions would be a polynomial of degree 6 and the number of formations for which there are three equal dimensions is a polynomial of degree 3.[1]

Thus for large values of t, the first term $t^9/1296$ completely dominates. Plugging in $t = 100$ yields $10^{18}/1296$, which is approximately 10^{15}.

[1] To count cases where the dimensions are equal, we must have the three last triples equal, say, (x,y,z). If the first triple is (a,b,c), we have $a+3x=t, b+3y=t, c+3z=t$. Each triple (a,b,c) corresponds to at most one formation, and hence the number of these formations is at most equal to the number of (a,b,c) triples which is equal to $(t+1)^3$.

For cases where there are two equal dimensions, we have the first triple (a,b,c) as before, and the next two equal triples (x,y,z) with a third triple (k,l,m). Our equations are now $a+2x+k=t, b+2y+l=t, c+2z+m=t$. Each choice of (a,b,c) triple, plus a choice of (k,l,m) triple would yield (at most) one formation. This is clearly at most a degree-6 polynomial in t, since the unrestricted choice of (a,b,c) is cubic, and the (k,l,m) triple depends on (a,b,c).