18th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 23, 2016

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems 1 and 2 on this page; problems 3, 4, 5 on other side.

1 The *distinct prime factors* of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5.

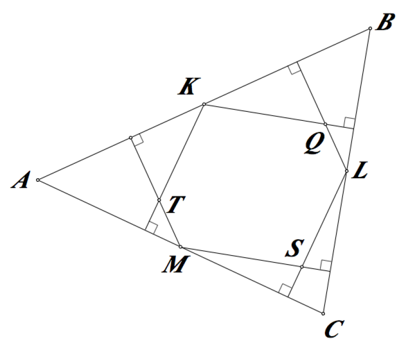
Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$).

Show that, for every integer k greater than or equal to 2,

(i) A and B have the same set of distinct prime factors.

(ii) $A+1$ and $B+1$ have the same set of distinct prime factors.

2 In an acute triangle ABC let K, L, and M be the midpoints of sides AB, BC, and CA, respectively. From each of K, L, and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and CA, etc. The resulting 6 perpendiculars intersect at points Q, S, and T as in the figure to form a hexagon $KQLSMT$ inside triangle ABC. Prove that the area of this hexagon $KQLSMT$ is half of the area of the original triangle ABC.



3 For $n > 1$, consider an $n \times n$ chessboard and place identical pieces at the centers of different squares.

(i) Show that no matter how $2n$ identical pieces are placed on the board, that one can always find 4 pieces among them that are the vertices of a parallelogram.

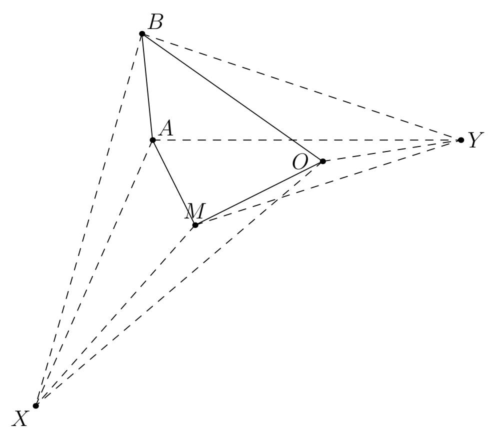
(ii) Show that there is a way to place $(2n-1)$ identical chess pieces so that no 4 of them are the vertices of a parallelogram.

4 Find a positive integer N and $a_1, a_2, \ldots, a_N$, where $a_k = 1$ or $a_k = -1$ for each $k = 1, 2, \ldots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_N \cdot N^3 = 20162016,$$

or show that this is impossible.

5 The corners of a fixed convex (but not necessarily regular) n-gon are labeled with distinct letters. If an observer stands at a point in the plane of the polygon, but outside the polygon, they see the letters in some order from left to right, and they spell a "word" (that is, a string of letters; it doesn't need to be a word in any language). For example, in the diagram below (where $n = 4$), an observer at point X would read "*BAMO*," while an observer at point Y would read "*MOAB*."



Determine, as a formula in terms of n, the maximum number of distinct n-letter words which may be read in this manner from a single n-gon. Do not count words in which some letter is missing because it is directly behind another letter from the viewer's position.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2016 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 2–4PM on Sunday, March 20 (note that this is a week later than last year). This event will include a mathematical talk by **Jacob Fox (Stanford University)**, refreshments, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor and/or `bamo.org` for a more detailed schedule, plus directions.