19th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 28, 2017

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems 1, 2, and 3 on this page; problems 4, 5 on the other side.

1 Find all positive integers n such that when we multiply all divisors of n, we will obtain 10^9. Prove that your number(s) n work and that there are no other such numbers.

(*Note:* A *divisor* of n is a positive integer that divides n without any remainder, including 1 and n. For example, the divisors of 30 are $1, 2, 3, 5, 6, 10, 15, 30$.)

2 The area of square $ABCD$ is $196\,\text{cm}^2$. Point E is inside the square, at the same distances from points D and C, and such that $\angle DEC = 150°$. What is the perimeter of $\triangle ABE$ equal to? Prove that your answer is correct.

3 Consider the $n \times n$ "multiplication table" below on the left. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table.

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

(1)	(2)	(3)	4	5
2	4	(6)	(8)	10
3	6	9	(12)	15
4	8	12	(16)	20
5	10	15	(20)	(25)

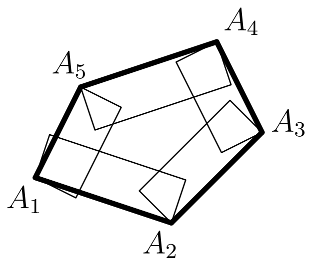
We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, in the case $n = 5$, one such possible path is shown above on the right, with all the numbers along the path circled. If we add up the circled numbers in the example above (including the start and end squares), we get 93. Considering all such possible paths on the $n \times n$ grid:

(a) What is the smallest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.

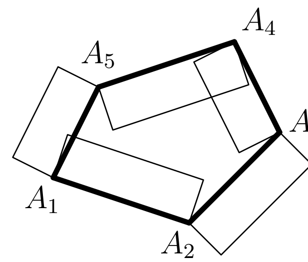
(b) What is the largest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.

4 Consider a convex n-gon $A_1A_2\cdots A_n$. (*Note*: In a convex polygon, all interior angles are less than 180°.) Let h be a positive number. Using the sides of the polygon as bases, we draw n rectangles, each of height h, so that each rectangle is either entirely inside the n-gon or partially overlaps the inside of the n-gon.

As an example, the left figure below shows a pentagon with a correct configuration of rectangles, while the right figure shows an incorrect configuration of rectangles (since some of the rectangles do not overlap with the pentagon):



(a) Correct



(b) Incorrect

Prove that it is always possible to choose the number h so that the rectangles completely cover the interior of the n-gon and the total area of the rectangles is no more than twice the area of the n-gon.

5 Call a number T *persistent* if the following holds: Whenever a,b,c,d are real numbers different from 0 and 1 such that

$$a+b+c+d=T$$

and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=T \ ,$$

we also have

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}+\frac{1}{1-d}=T \ .$$

(a) If T is persistent, prove that T must be equal to 2.

(b) Prove that 2 is persistent.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2017 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 2–4PM on Sunday, March 12 (note that this is a week earlier than last year). This event will include a mathematical talk by **Matthias Beck (San Francisco State University)**, refreshments, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor and/or `bamo.org` for a more detailed schedule, plus directions.