20th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 27, 2018

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems 1 and 2 are on this side; problems 3, 4, and 5 are on the other side.

1 An integer c is *square-friendly* if it has the following property: For every integer m, the number $m^2+18m+c$ is a perfect square. (A perfect square is a number of the form n^2, where n is an integer. For example, $49 = 7^2$ is a perfect square while 46 is not a perfect square. Further, as an example, 6 is not square-friendly because for $m = 2$, we have $(2)^2 + (18)(2) + 6 = 46$, and 46 is not a perfect square.)

In fact, exactly one square-friendly integer exists. Show that this is the case by doing the following:

(a) Find a square-friendly integer, and prove that it is square-friendly.

(b) Prove that there cannot be two different square-friendly integers.

2 Let points P_1, P_2, P_3, and P_4 be arranged around a circle in that order. (One possible example is drawn in Diagram 1.) Next draw a line through P_4 parallel to $\overline{P_1P_2}$, intersecting the circle again at P_5. (If the line happens to be tangent to the circle, we simply take $P_5 = P_4$, as in Diagram 2. In other words, we consider the second intersection to be the point of tangency again.) Repeat this process twice more, drawing a line through P_5 parallel to $\overline{P_2P_3}$, intersecting the circle again at P_6, and finally drawing a line through P_6 parallel to $\overline{P_3P_4}$, intersecting the circle again at P_7. Prove that P_7 is the same point as P_1.

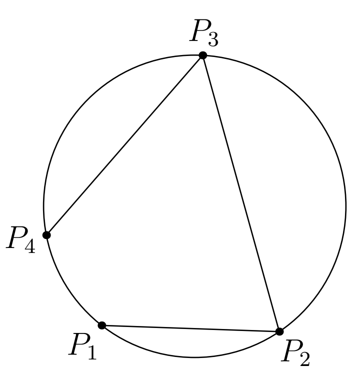


Diagram 1

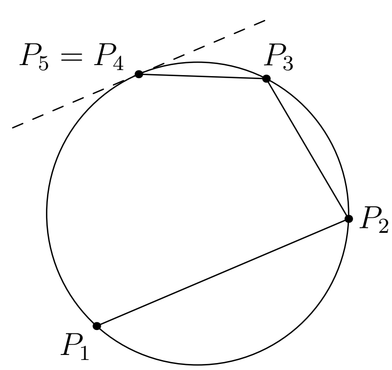


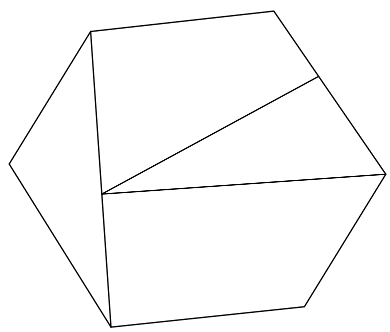
Diagram 2

3 Suppose that 2002 numbers, each equal to 1 or -1, are written around a circle. For every two adjacent numbers, their product is taken; it turns out that the sum of all 2002 such products is negative. Prove that the sum of the original numbers has absolute value less than or equal to 1000. (The absolute value of x is usually denoted by $|x|$. It is equal to x if $x \geq 0$, and to $-x$ if $x < 0$. For example, $|6| = 6$, $|0| = 0$, and $|-7| = 7$.)

4 (a) Find two quadruples of positive integers (a,b,c,n), each with a different value of n greater than 3, such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=n$.

(b) Show that if a,b,c are nonzero integers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ is an integer, then abc is a perfect cube. (A perfect cube is a number of the form n^3, where n is an integer.)

5 To *dissect* a polygon means to divide it into several regions by cutting along finitely many line segments. For example, the diagram below shows a dissection of a hexagon into two triangles and two quadrilaterals:



An *integer-ratio right triangle* is a right triangle whose side lengths are in an integer ratio. For example, a triangle with sides $3,4,5$ is an integer-ratio right triangle, and so is a triangle with sides $\frac{5}{2}\sqrt{3}, 6\sqrt{3}, \frac{13}{2}\sqrt{3}$. On the other hand, the right triangle with sides $\sqrt{2}, \sqrt{5}, \sqrt{7}$ is not an integer-ratio right triangle.

Determine, with proof, all integers n for which it is possible to completely dissect a regular n-sided polygon into integer-ratio right triangles.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2018 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 2–4PM on Sunday, March 11 . This event will include a mathematical talk by **Tadashi Tokieda (Stanford)**, refreshments, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor and/or `bamo.org` for a more detailed schedule, plus directions.