21st Bay Area Mathematical Olympiad

BAMO-12 Exam

February 26, 2019

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

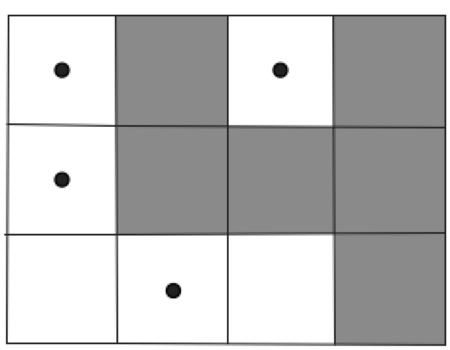
The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems 1 and 2 are on this side; problems 3, 4, 5 are on the other side.

1 You are traveling in a foreign country whose currency consists of five different-looking kinds of coins. You have several of each coin in your pocket. You remember that the coins are worth 1, 2, 5, 10, and 20 florins, but you have no idea which coin is which and you don't speak the local language. You find a vending machine where a single candy can be bought for 1 florin: you insert any kind of coin, and receive 1 candy plus any change owed. You can only buy one candy at a time, but you can buy as many as you want, one after the other.

What is the least number of candies that you must buy to ensure that you can determine the values of all the coins? Prove that your answer is correct.

2 Initially, all the squares of an 8×8 grid are white. You start by choosing one of the squares and coloring it gray. After that, you may color additional squares gray one at a time, but you may only color a square gray if it has exactly 1 or 3 gray neighbors at that moment (where a neighbor is a square sharing an edge). For example, the configuration below (of a smaller 3×4 grid) shows a situation where six squares have been colored gray so far. The squares that can be colored at the next step are marked with a dot.



Is it possible to color all the squares gray? Justify your answer.

3 In triangle $\triangle ABC$, we have marked points A_1 on side BC, B_1 on side AC, and C_1 on side AB so that AA_1 is an altitude, BB_1 is a median, and CC_1 is an angle bisector. It is known that $\triangle A_1B_1C_1$ is equilateral. Prove that $\triangle ABC$ is equilateral too.

(*Note:* A median connects a vertex of a triangle with the midpoint of the opposite side. Thus, for median BB_1 we know that B_1 is the midpoint of side AC in $\triangle ABC$.)

4 Let S be a finite set of **nonzero** real numbers, and let $f : S \to S$ be a function with the following property: for each $x \in S$, either

$$f(f(x)) = x + f(x) \qquad \text{or} \qquad f(f(x)) = \frac{x + f(x)}{2}.$$

Prove that $f(x) = x$ for all $x \in S$.

5 Every positive integer is either *nice* or *naughty*, and the Oracle of Numbers knows which are which. However, the Oracle will not directly tell you whether a number is nice or naughty. The only questions the Oracle will answer are questions of the form "What is the sum of all nice divisors of n?," where n is a number of the questioner's choice. For instance, suppose (*just* for this example) that 2 and 3 are nice, while 1 and 6 are naughty. In that case, if you asked the Oracle, "What is the sum of all nice divisors of 6?," the Oracle's answer would be 5.

Show that for any given positive integer n less than 1 million, you can determine whether n is nice or naughty by asking the Oracle at most four questions.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2019 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 2–4PM on Sunday, March 10. This event will include a mathematical talk by **Mira Bernstein (Tufts University)**, refreshments, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor and/or `bamo.org` for a more detailed schedule, plus directions.