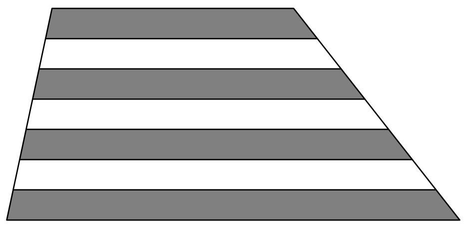
# 22nd Bay Area Mathematical Olympiad

## Problems and Solutions

February 26, 2020

*The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.*

**A** A trapezoid is divided into seven strips of equal width as shown. What fraction of the trapezoid's area is shaded? Explain why your answer is correct.



**Solution.** We can fit the trapezoid together with a rotated copy of itself to make a parallelogram:

The stripes on the two trapezoids join to make seven stripes across the parallelogram. These stripes are congruent to each other, so the shaded ones cover exactly $\frac{4}{7}$ of the area of the parallelogram. Since the two copies of the original trapezoid are identical, the stripes must also cover $\boxed{\frac{4}{7}}$ of the area of each trapezoid. ■

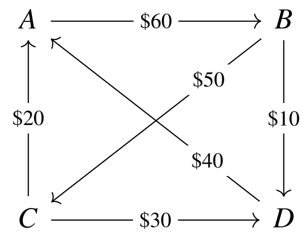
**B** Four friends, Anna, Bob, Celia, and David, exchanged some money. For any two of these friends, exactly one gave some money to the other. For example, Celia could have given money to David but then David would not have given money to Celia. In the end, each person broke even (meaning that no one made or lost any money).

(a) Is it possible that the amounts of money given were \$10, \$20, \$30, \$40, \$50, and \$60?

(b) Is it possible that the amounts of money given were \$20, \$30, \$40, \$50, \$60, and \$70?

For each part, if your answer is yes, show that the situation is possible by describing who could have given what amounts to whom. If your answer is no, prove that the situation is not possible.

**Solution.** The answer to (a) is yes. For example, the transactions could be as follows:




The answer to (b) is no. Let's reason by contradiction: assume that it is possible for the four friends to exchange the amounts \$20, \$30, \$40, \$50, \$60, and \$70. Without loss of generality, assume that A gave \$60 to B. Then A must have received payments of \$20 and \$40 (since $60 = 20 + 40$ is the only way to write 60 as a sum or difference of two numbers among 20, 30, 40, 50, and 70). By the exact same reasoning, B must have given away \$20 and \$40; and yet those amounts were received by A. Hence, B must have given \$20 or \$40 to A, contradicting that A gave B money. This shows that our assumption is false, and that it is not possible for the four friends to exchange the amounts \$20, \$30, \$40, \$50, \$60, and \$70. ■

**C/1** Find all real numbers $x$ that satisfy the equation

$$\frac{x-2020}{1}+\frac{x-2019}{2}+\cdots+\frac{x-2000}{21}=\frac{x-1}{2020}+\frac{x-2}{2019}+\cdots+\frac{x-21}{2000},$$

and simplify your answer(s) as much as possible. Justify your solution.

**Solution 1.** The number $x = 2021$ works. Indeed, for $x = 2021$, the left-hand side of the equation equals

$$\frac{2021-2020}{1}+\frac{2021-2019}{2}+\cdots+\frac{2021-2000}{21}=\frac{1}{1}+\frac{2}{2}+\cdots+\frac{21}{21}=\underbrace{1+1+\cdots+1}_{21}=21,$$

and the right-hand side of the equation equals the same number:

$$\frac{2021-1}{2020}+\frac{2021-2}{2019}+\cdots+\frac{2021-21}{2000}=\frac{2020}{2020}+\frac{2019}{2019}+\cdots+\frac{2000}{2000}=\underbrace{1+1+\cdots+1}_{21}=21.$$

Why is $x = 2021$ the only solution? The equation is linear: after simplifying it, it can be put into the form $ax+b=0$. Such equations have a unique solution, namely, $x=-\frac{b}{a}$, as long as the coefficient $a$ of $x$ is not 0. In our situation, $x$ is multiplied by

$$a=\underbrace{\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{21}\right)}_{S_1}-\underbrace{\left(\frac{1}{2020}+\frac{1}{2019}+\cdots+\frac{1}{2000}\right)}_{S_2}.$$

However, each of the 21 fractions in the first sum $S_1$ is bigger than each of the 21 fractions in the second sum $S_2$. Thus, $S_1 > S_2$ and $a > 0$. Since $a \neq 0$, the equation has a *unique* solution, which we found earlier to be $x = 2021$. ■

**Solution 2.** If we subtract 21 from both sides (i.e., we subtract 1 from each of the 21 fractions on both sides), we obtain

$$\frac{x-2021}{1}+\frac{x-2021}{2}+\cdots+\frac{x-2021}{21}=\frac{x-2021}{2020}+\frac{x-2021}{2019}+\cdots+\frac{x-2021}{2000}$$

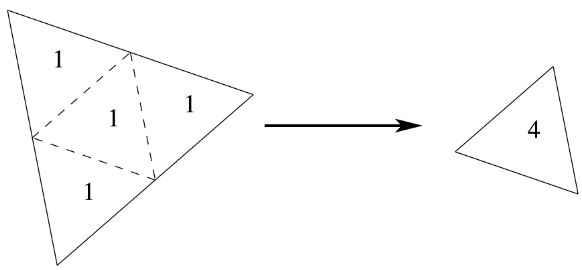
or

$$\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{21}\right)(x-2021)=\left(\frac{1}{2020}+\frac{1}{2019}+\cdots+\frac{1}{2000}\right)(x-2021)$$

which implies $x-2021=0$, or $x=2021$. ■

**Remark.** The problem is adapted from Regalia-6 publ., Sofia, Bulgaria.

**D/2** Consider a sheet of paper in the shape of an equilateral triangle creased along the dashed lines as in the figure below on the left. Folding over each of the three corners along the dashed lines creates a new object which is uniformly four layers thick, as in the figure below on the right. The number in each region indicates that region's thickness (in layers of paper).




We have just seen one example of how a plane figure can be folded into an object with a uniform thickness. This problem asks you to produce several other examples. In each case, you may fold along any lines. The different parts that are folded may or may not be congruent. Assume that paper may be folded any number of times without tearing or becoming too thick to fold. If needed, you can use any of the following tools:

- a magic ruler with which you can draw a line through any two given points and you can split any segment into as many equal parts as you wish; and
- a right triangle tool with which you can drop perpendiculars from points to lines and erect perpendiculars to lines from points on them.

Given these rules:

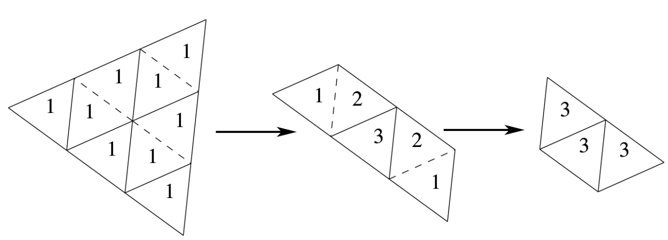
(a) Show how to fold an equilateral triangle into an object with a uniform thickness of 3 layers.

(b) Show how to fold a 30°-60°-90° triangle into an object with a uniform thickness of 3 layers.

(c) Show that every triangle can be folded into an object with a uniform thickness of 2020 layers.

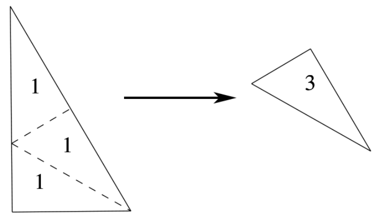
**Solution.**

(a) Divide the sides of the triangle into three equal parts and connect them with lines as in the figure on the left below.



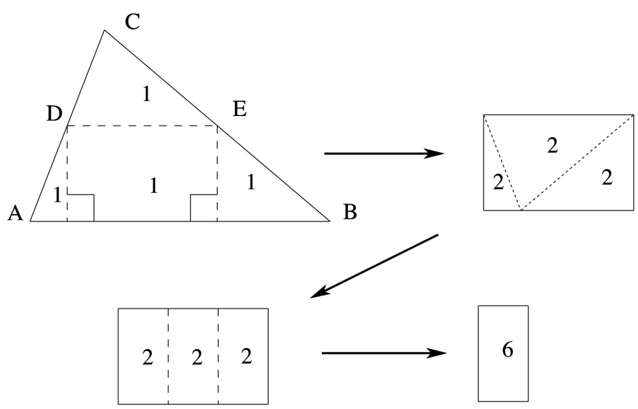
Now crease and fold along the dotted lines to form the figure in the center. Finally, fold along the dashed lines in the center figure to form the figure on the right. ■

(b) Divide the triangle into three triangles as shown in the figure on the left below. (The upper dashed line is the perpendicular bisector of the hypotenuse and its intersection with the vertical edge is connected to the lower-right vertex of the triangle.)




Now crease and fold along the two dashed lines to achieve the figure on the right. ■

(c) In fact, any triangle can be folded to a uniform thickness of $2n$ layers for any positive integer $n$. Label the vertices of the triangle $A, B, C$ so that $AB$ is the longest side, guaranteeing that $\angle A$ and $\angle B$ are acute. Let points $D$ and $E$ bisect the segments $AC$ and $BC$. Next drop perpendiculars from $D$ and $E$ to $AB$ and fold along the three dashed lines to form the rectangle in the upper right of the figure below.




That rectangle is 2 layers thick. To make an object of $2n$ layers, divide the rectangle's top and bottom edges into $n$ equal pieces and connect pairs of points to make $n$ regions. Folding along all $n-1$ dashed lines will produce an object with $2n$ layers. The figure demonstrates how to do this if $n = 3$, yielding an object with $2n = 6$ layers. For 2020 layers, let $n = 1010$. ■

**E/3** The integer 202020 is a multiple of 91. For every positive integer $n$, show how $n$ additional 2's may be inserted into the digits of 202020 so that the resulting $(n+6)$-digit integer is also a multiple of 91.

For example, a possible way to do this when $n = 5$ is $\underline{2}2020\underline{2}20\underline{222}$ (the inserted 2's are underlined).

**Solution.** Every integer of the form $202\ldots2020$ (where the dots represent any number of 2's) is divisible by 91. There are a variety of ways to discover and prove this fact, some of which we outline here:

*Method 1.* For $n = 1$, there are four options: 2202020, 2022020, 2020220, and 2020202. Long division shows that only 2022020 is divisible by 91. Let $a_0 = 202020$ and $a_1 = 2022020$. Then $a_1 - a_0 = 1820000$. Clearly, adding $182 \cdot 10^k$ for some $k \geq 0$ preserves divisibility by 91. From this, we conjecture that recursively defining $a_n = a_{n-1} + 182 \cdot 10^{n+3}$ yields integers of the desired form. We let $\overset{2}{\ldots}$ denote a string of $n$ twos and $\overset{0}{\ldots}$ denote a string of $n$ zeros. If $a_{n-1} = 20\overset{2}{\ldots}020$, then

$$a_n = a_{n-1} + 182 \cdot 10^{n+3} = 20\overset{2}{\ldots}020 + 182\overset{0}{\ldots}000 = 202\overset{2}{\ldots}020.$$

By induction, it follows that $a_n$ is a multiple of 91 of the form $202\ldots2020$, for any $n \geq 0$.

■

*Method 2.* As in (a), we can see that the only solution for $n = 1$ is 2022020. Let $a_0 = 202020$ and 2022020. Then $a_1 - 10a_0 = 1820$. Clearly, adding 1820 preserves divisibility by 91. We conjecture that recursively defining $a_n = 10a_{n-1} + 1820$ yields integers of the desired form. If $a_{n-1} = 202\ldots2020$, then

$$a_n = 10a_{n-1} + 1820 = 10 \cdot 202\ldots2020 + 1820 = 202\cdots20200 + 1820 = 202\ldots22020.$$

By induction, it follows that $a_n$ is a multiple of 91 of the form $202\ldots2020$, for any $n \geq 0$.

■

*Method 3.* Long division or factoring yields $202020 \div 91 = 2220$, so we conjecture that $22220 \times 91$, $222220 \times 91$, etc. will have a nice pattern of digits. Indeed, these numbers are of the form $202\cdots2020$ (where the dots represent 2's), and we can explain why with long arithmetic:

$$\begin{array}{rrcrcrr} & 222\cdots20 & \times & 100 & = & & 222\cdots2000 \\ - & 222\cdots20 & \times & 10 & = & - & 22\cdots2200 \\ + & 222\cdots20 & \times & 1 & = & + & 2\cdots2220 \\ \hline & 222\cdots20 & \times & 91 & = & & 202\cdots2020 \end{array}$$

(all ellipses represent strings of 2's which may be made any length). ■

*Method 4.* As in (a), we see that the only solution for $n = 1$ is 2022020. Let $\overset{2}{\ldots}$ denote a string of $n$ twos, for any $n \geq 0$. We conjecture that $a_n = 20\overset{2}{\ldots}020$ is divisible by 91 for any $n \geq 0$. We can check that $a_1 = 202020$ and $a_0 = 2002$ are divisible by 91. Since $1000 \equiv -1 \pmod{91}$, we have

$$\begin{aligned} a_{n+2} &= 20\overset{2}{\ldots}22020 = 1000 \cdot 20\overset{2}{\ldots}22 + 20 \\ &\equiv 1000 \cdot (20\overset{2}{\ldots}22 - 20) = 1000 \cdot 20\overset{2}{\ldots}02 \\ &= 100 \cdot 20\overset{2}{\ldots}020 = 100 \cdot a_n \pmod{91}. \end{aligned}$$

But $\gcd(100, 91) = 1$, so if $91 \mid a_n$, then $91 \mid a_{n+2}$. By induction, we have $91 \mid a_n$ for all $n \geq 0$. ■

*Method 5.* As in (a), we see that the only solution for $n = 1$ is 2022020. Let $\overset{2}{\ldots}$ denote a string of $n$ twos, for $n \geq 0$. Notice that $-10 \cdot 9 \equiv 1$ and $10^3 \equiv -1 \pmod{91}$. Therefore,

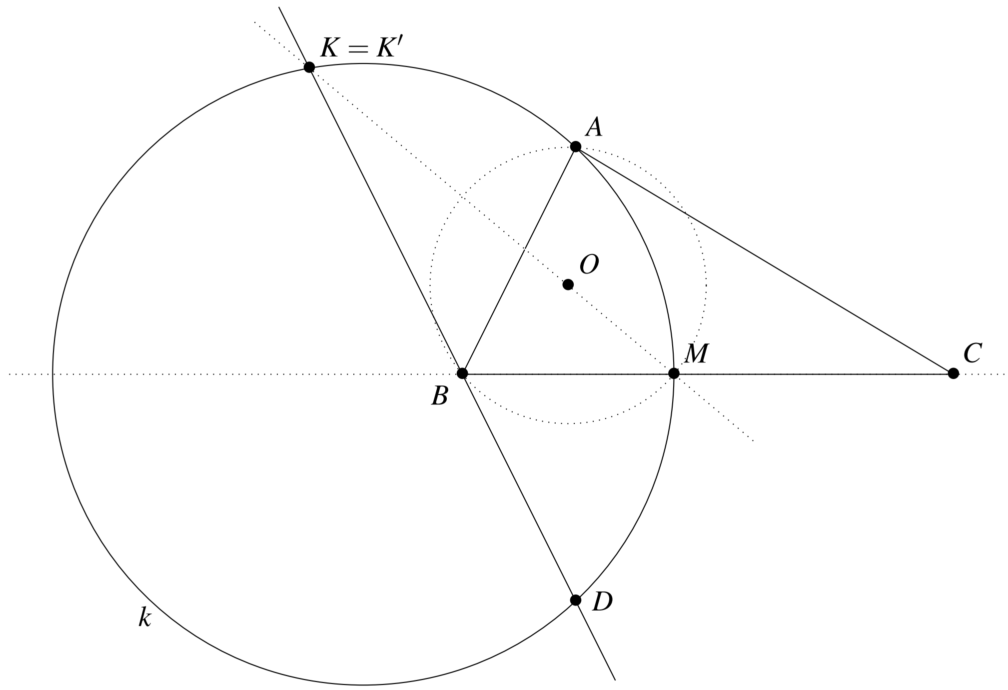
$$\begin{aligned} 20\overset{2}{\ldots}020 &= 22\overset{2}{\ldots}222 - 2 \cdot 10^{k+3} - 202 \\ &\equiv \tfrac{2}{9} \cdot 99\ldots999 + 2 \cdot 10^k - 20 \\ &\equiv -20 \cdot (10^{k+5} - 1) + 2 \cdot 10^k - 20 \\ &= -2 \cdot 10^{k+6} + 20 + 2 \cdot 10^k - 20 \\ &\equiv -2 \cdot 10^k + 2 \cdot 10^k = 0 \pmod{91}. \end{aligned}$$

■

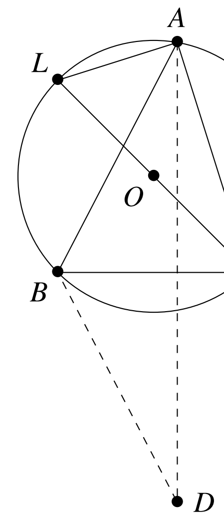
**Remark.** A BAMO contestant solved this problem by proving the polynomial identity $(x^2 - x + 1)(x^k + x^{k-1} + \cdots + x + 1) = x^{k+2} + (x^k + x^{k-1} + x^{k-2} + \cdots + x^4 + x^3 + x^2) + 1$, valid for each integer $k \geq 2$. Substituting $x = 10$ leads to a solution similar to Method 3 above, but other values may be substituted for $x$ to show an analogous result in any number base.

**4** Consider $\triangle ABC$. Choose a point $M$ on its side $BC$ and let $O$ be the center of the circle passing through the vertices of $\triangle ABM$. Let $k$ be the circle that passes through $A$ and $M$ and whose center lies on line $BC$. Let line $MO$ intersect $k$ again in point $K$. Prove that the line $BK$ is the same for any choice of point $M$ on segment $BC$, so long as all of these constructions are well-defined.

**Solution.** Let $D$ be the reflection of $A$ across side $BC$, which clearly lies on $k$. Let $K'$ be the point where lines $BD$ and $MO$ intersect. We will eventually show that $K' = K$. Then $K$ lies on line $BD$, which is therefore the same as line $BK$. Since $B$ and $D$ don't depend on the choice of $M$, this proves that the line $BK$ does not depend on $M$, if all of the constructions are well-defined. One possible configuration is illustrated below. Notice that we cannot have $M = B$, because then the circumcenter $O$ of triangle $\triangle ABM$ would not be well-defined (we will need this later).



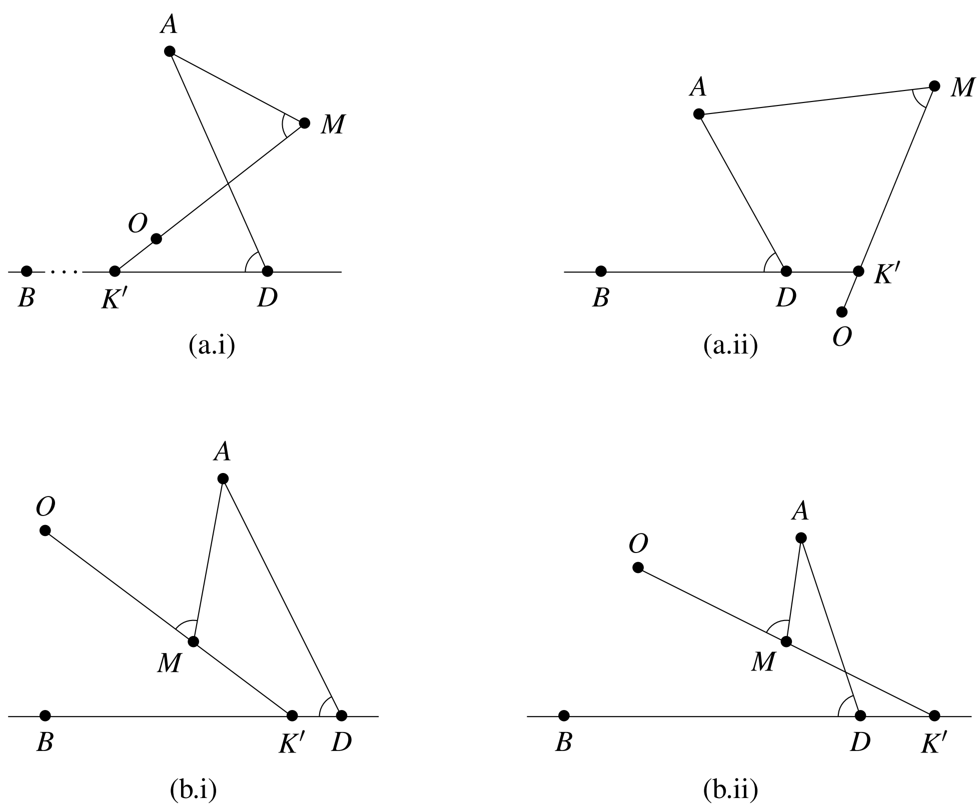

We first show that $\angle AMO = \angle BDA$, by focusing on the circumcircle of $\triangle ABM$, as illustrated below. Let $L$ be the point diametrically opposite $M$. Then $\angle LAM$ is right and thus $\angle AMO = 90^\circ - \angle ALM$. Since $AD \perp BM$, we have $\angle BDA = \angle BAD = 90^\circ - \angle ABM$. But the angles $\angle ABM$ and $\angle ALM$ subtend the same arc, so $\angle ABM = \angle ALM$ and thus $\angle BDA = \angle AMO$, as desired.




We will now show that the points $A$, $D$, $M$, and $K'$ are concyclic. Recall the following theorem.

> A convex quadrilateral $EFGH$ is cyclic if either of the following are true:
> (1) $\angle EFG + \angle EHG = 180^\circ$. (2) $\angle EGF = \angle EHF$.

We will apply this theorem to four separate cases, illustrated below.




(a) Suppose that $M$ lies to the right of $AD$. Then $O$ and $K'$ are on the same side of $M$.

i. If $B$ and $K'$ are on the same side of $D$, then $\angle AMK' = \angle AMO = \angle ADB = \angle ADK'$. Therefore, quadrilateral $AMDK'$ is cyclic by condition (2).

ii. If $B$ and $K'$ are on opposite sides of $D$, then $\angle AMK' = \angle AMO = \angle ADB = 180^\circ - \angle ADK'$, so $\angle AMK' + \angle ADK' = 180^\circ$. Therefore, quadrilateral $AMK'D$ is cyclic by condition (1).

(b) Now suppose that $M$ lies to the left of $AD$. Then $O$ and $K'$ are on opposite sides of $M$.

i. If $B$ and $K'$ are on the same side of $D$, then $\angle ADK' = \angle ADB = \angle AMO = 180^\circ - \angle AMK'$, so $\angle ADK' + \angle AMK' = 180^\circ$. Therefore, quadrilateral $ADK'M$ is cyclic by condition (1).

ii. If $B$ and $K'$ are on opposite sides of $D$, then $\angle ADK' = 180^\circ - \angle ADB = 180^\circ - \angle AMO = \angle AMK'$. Therefore, quadrilateral $AK'DM$ is cyclic by condition (2).

This proves that the points $A$, $D$, $M$, and $K'$ are concyclic. Since $k$ is the unique circle going through points $A$, $D$, and $M$, it follows that $K'$ lies on $k$. Thus $K'$ lies on line $MO$ and circle $k$, so we have $K' = K$ or $K' = M$. But if $K' = M$, then $M$ lies on line $BD$ and thus $M = B$. We noted above that this cannot be the case (or $O$ will not be well-defined), so we have $K = K'$, as desired. ■

**Remark.** Matyas Sustik points out that $M$ is the center of the in- or excircle of $\triangle ABK$; a solution can be constructed on this basis.

**5** Let $S$ be a set of $a+b+3$ points on a sphere, where $a,b$ are nonnegative integers and no four points of $S$ are coplanar (that is, no four points lie on a plane). Determine how many planes pass through three points of $S$ and separate the remaining points into $a$ points on one side of the plane and $b$ points on the other side.

**Solution.** Let $f(a,b)$ be the number of oriented planes through three of the given $a+b+3$ points, such that exactly $a$ points lie above the plane (a plane divides 3D-space into two regions; an oriented plane can be thought of as a plane and a choice of one side "above" the plane). We then recover $g$ as

$$g(a,b) = \begin{cases} f(a,b), & a \neq b \\ \frac{1}{2} f(a,b), & a = b \end{cases}$$

because each plane is counted twice when $a = b$.

We begin by calculating $f(a,0)$. Consider the convex hull $\Sigma$ of our $a+3$ points. This is a polyhedron, so we apply Euler's formula $V - E + F = 2$. Since our $a+3$ points lie on a sphere, they are all vertices of $\Sigma$ and hence $V = a+3$. No four of our points are coplanar, so each face of $\Sigma$ is a triangle and thus $2E = 3F$. Therefore,

$$F = 3F - 2F = 2E - 2(2 - V + E) = 2(V - 2) = 2(a+1).$$

But a plane through three of our points has all the other points on one side iff it does not cut through the interior of $\Sigma$, i.e. iff it is a face of $\Sigma$ (the orientation should be such that the other points lie above the plane of this face). Hence $f(a,0) = F = 2(a+1)$.

Next, notice that $f(a,b) = f(b,a)$ for any $a,b \geq 0$, because reversing the orientations of all planes interchanges the roles of $a$ and $b$. Moreover, if $n = a+b+3$, then there are a total of $2\binom{n}{3}$ oriented planes passing through three of our points, so $f(n-3,0) + f(n-4,1) + \cdots + f(1,n-4) + f(0,n-3) = 2\binom{n}{3}$. Some small values of $f(a,b)$ can now be calculated directly:

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 2 | 4 | 6 | 8 |
| 1 | 4 | 8 | 12 | |
| 2 | 6 | 12 | | |
| 3 | 8 | | | |

Based on these small cases, we conjecture that

$$f(a,b) = 2(a+1)(b+1). \tag{$*$}$$

We will prove this by induction.

Consider a fixed arrangement of $n+1$ points (on a sphere, with no four points coplanar) and suppose that $(*)$ holds for any $n$ points (the base case of $n=3$ is trivial). Consider $a,b \geq 0$ with $a+b+3=n$. Let $S_{a,b}$ denote the set of all pairs $(P,\Pi)$, where $P$ is one of our $n+1$ points, $\Pi$ is an oriented plane through three of the other points, and precisely $a$ of the remaining points lie above $\Pi$. By first choosing $P$ and then $\Pi$, we find that

$$|S_{a,b}| = (n+1)f(a,b) = 2(n+1)(a+1)(b+1),$$

where the second equality is by the induction hypothesis. By first choosing $\Pi$ and then $P$, we find that

$$|S_{a,b}| = (a+1)f(a+1,b) + (b+1)f(a,b+1).$$

Equating these two expressions for $|S_{a,b}|$, we have

$$(a+1)f(a+1,b) + (b+1)f(a,b+1) = 2(a+b+4)(a+1)(b+1), \tag{1}$$

for any $a,b \geq 0$ such that $a+b+3=n$. But if we have $f(a+1,b) = 2(a+2)(b+1)$, then (1) gives

$$\begin{aligned}(b+1)f(a,b+1) &= 2(a+b+4)(a+1)(b+1) - (a+1)f(a+1,b)\\ &= 2(a+b+4)(a+1)(b+1) - 2(a+1)(a+2)(b+1)\\ &= 2(a+1)(b+1)\big((a+b+4)-(b+2)\big) = 2(a+1)(b+1)(b+2),\end{aligned}$$

and thus $f(a,b+1) = 2(a+1)(b+2)$. Thus, for any $a,b \geq 0$ with $a+b+3=n$,

$$f(a+1,b) \text{ satisfies } (*) \quad \Longrightarrow \quad f(a,b+1) \text{ satisfies } (*).$$

Starting with the base case $f(n-2,0) = n-1$ (proven above) and applying another layer of induction, we see that the values $f(n-2,0),\ f(n-3,1),\ f(n-4,2),\ \ldots,\ f(0,n-2)$ all satisfy $(*)$. This proves the induction hypothesis (i.e. that $(*)$ holds for any $n+1$ points) and thus concludes the proof. ∎